General Relativity Fall 2018
Lecture 5: the metric tensor field

Yacine Ali-Haïmoud
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Definition and basic properties – The metric $g_{\alpha\beta}$ is a rank (0, 2) symmetric tensor field, that gives the notion of *norms* and *angles* on the tangent space, and allows to compute infinitesimal "distances". It is *non-degenerate*, i.e. such that $g_{\alpha\beta}X^\beta = 0 \Rightarrow X^\alpha = 0$. The components of a metric are often given through the *line element* $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$. The *inverse metric* $g^{\alpha\beta}$ is such that $g_{\alpha\beta}g^{\beta\gamma} = \delta_\alpha{}^\gamma$. The symmetry of $g_{\alpha\beta}$ implies it has $n(n+1)/2$ independent components. In dimension $n = 4$, this means 10 independent components. The metric can be used to lower indices, i.e. to define a dual vector $X_\alpha \equiv g_{\alpha\beta}X^\beta$ given a vector X^α. Similarly, the inverse metric can be used to raise indices, i.e. define a vector given a dual vector.

Normal form – Such a symmetric, non-degenerate tensor can always be put in *normal form* under a linear change of coordinates (see homework 2), i.e. diagonalized, with diagonal elements all equal to ± 1. The number of pluses and minuses is independent of the basis in which the metric is diagonal (this theorem is called Sylvester's law of inertia). This number of pluses and minuses is called the *signature* of the metric. We will work with a metric of signature (-, +,+,+). In other words, at any given point p of the manifold, we can always find a linear transformation of coordinates such that $g_{\mu\nu}|_p = \eta_{\mu\nu}$ in these coordinates, where $\eta_{\mu\nu}$ is the Minkowski metric. These coordinates are defined up to a Lorentz transformation, which, by definition, leaves the Minkowski metric unchanged.

Locally inertial coordinates (LICs) – We can actually do better than the normal form: at any point p, we can find local coordinates (defined in a neighborhood of p) in which $g_{\mu\nu}|_p = \eta_{\mu\nu}$ *and* $\partial_\lambda g_{\mu\nu}|_p = 0$, i.e. such that the metric is Minkowski up to deviations quadratic in the coordinates. In general, however, we *cannot* set the second derivatives to zero. Let us prove this. Under a change of coordinates, the metric components change as

$$g_{\mu'\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^\nu}{\partial x^{\nu'}}g_{\mu\nu}. \tag{1}$$

Now suppose that we want $g_{\mu\nu}|_p = \eta_{\mu\nu}$. This implies

$$\frac{\partial x^\mu}{\partial x^{\mu'}}\Big|_p\frac{\partial x^\nu}{\partial x^{\nu'}}\Big|_p\eta_{\mu\nu} = g_{\mu'\nu'}|_p. \tag{2}$$

These are $n(n+1)/2 = 10$ independent equations (the 10 independent components of $g_{\mu'\nu'}$) for the $n^2 = 16$ variables $\partial x^\mu/\partial x^{\mu'}|_p$. So clearly, we should have more than enough freedom to set $g_{\mu\nu}|_p$: we have $n(n-1)/2 = 6$ extra degrees of freedom, which correspond to Lorentz transformations (3 rotations and 3 boosts).

Let us now take the derivative of Eq. (**??**):

$$\frac{\partial g_{\mu'\nu'}}{\partial x^{\lambda'}} = \frac{\partial}{\partial x^{\lambda'}}\left(\frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^\nu}{\partial x^{\nu'}}g_{\mu\nu}\right) = \frac{\partial}{\partial x^{\lambda'}}\left(\frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^\nu}{\partial x^{\nu'}}\right)g_{\mu\nu} + \frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^\nu}{\partial x^{\nu'}}\frac{\partial x^\lambda}{\partial x^{\lambda'}}\frac{\partial g_{\mu\nu}}{\partial x^\lambda}, \tag{3}$$

where we used the chain rule for the second term, $\partial/\partial x^{\lambda'} = (\partial x^\lambda/\partial x^{\lambda'})\partial/\partial x^\lambda$. If we now ask that $g_{\mu\nu}|_p = \eta_{\mu\nu}$ *and* $\partial_\lambda g_{\mu\nu}|_p = 0$, we find

$$\frac{\partial}{\partial x^{\lambda'}}\left(\frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^\nu}{\partial x^{\nu'}}\right)\Big|_p\eta_{\mu\nu} = \left(\frac{\partial^2 x^\mu}{\partial x^{\mu'}\partial x^{\lambda'}}\frac{\partial x^\nu}{\partial x^{\nu'}} + \frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial^2 x^\nu}{\partial x^{\nu'}\partial x^{\lambda'}}\right)\Big|_p\eta_{\mu\nu} = \frac{\partial g_{\mu'\nu'}}{\partial x^{\lambda'}}\Big|_p. \tag{4}$$

This gives us $n \times n(n+1)/2 = 40$ equations, for the $n \times (n+1)/2 = 40$ second derivatives $\partial^2 x^\nu/\partial x^{\mu'}\partial x^{\lambda'}|_p$. Therefore, we have just enough freedom in the coordinates to set the first derivative of the metric to zero.

Finally, taking one more derivative, we get

$$\frac{\partial^2 g_{\mu'\nu'}}{\partial x^{\lambda'}\partial x^{\sigma'}} = \frac{\partial^2}{\partial x^{\lambda'}\partial x^{\sigma'}}\left(\frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^\nu}{\partial x^{\nu'}}\right)g_{\mu\nu} + \frac{\partial}{\partial x^{\lambda'}}\left(\frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^\nu}{\partial x^{\nu'}}\right)\frac{\partial x^\sigma}{\partial x^{\sigma'}}\frac{\partial g_{\mu\nu}}{\partial x^\sigma} + \frac{\partial}{\partial x^{\sigma'}}\left(\frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^\nu}{\partial x^{\nu'}}\frac{\partial x^\lambda}{\partial x^{\lambda'}}\right)\frac{\partial g_{\mu\nu}}{\partial x^\lambda} + \frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^\nu}{\partial x^{\nu'}}\frac{\partial x^\lambda}{\partial x^{\lambda'}}\frac{\partial x^\sigma}{\partial x^{\sigma'}}\frac{\partial^2 g_{\mu\nu}}{\partial x^\lambda\partial x^\sigma}. \tag{5}$$

If we wanted to further impose that the second derivative of the metric vanishes at p, we would have $[n(n+1)/2]^2$ equations, for the $n \times n(n+1)(n+2)/6$ third derivatives $\partial^3 x^\nu / \partial x^{\mu'} \partial x^{\lambda'} \partial x^{\nu'}|_p$ – the number comes from the symmetry of the partial derivatives. That means we have $[n(n+1)/2]^2 - n(n+1)(n+2)/6 = n^2(n^2-1)/12$ too few variables. This is precisely the number of independent components of the Riemann tensor, which we will define and study later on, and which is a measure of the *curvature* of spacetime. In dimension 4, this is 20 independent components.

Physical interpretation of LICs – The fact that the first derivatives of the metric can be cancelled is simply a mathematical rephrasing of the equivalence principle: constant and uniform gravitational fields can always be eliminated by appropriate changes of coordinates, and cannot be measured by any means. The first derivatives of the metric play the role of the local gravitational field. The second derivatives, on the other hand, cannot be eliminated: they represent gravitational *tides*, which indeed *can* be measured.