Exercise 1: Parallel transport on a sphere

The 2-sphere of radius \( r \) has line element \( ds^2 = r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2 \).

(i) Compute the Christoffel symbols of this metric.

The inverse metric is diagonal, with components \( g^{\theta \theta} = 1/r^2 \), \( g^{\varphi \varphi} = 1/(r \sin \theta)^2 \), \( g^{\theta \varphi} = 0 \).

There are only 6 independent Christoffel symbols to compute:

\[
\Gamma^\theta_{\theta \theta} = \frac{1}{2r^2} g^{\theta \theta, \theta} = 0, \tag{1}
\]

\[
\Gamma^\theta_{\theta \varphi} = \frac{1}{2r^2} g^{\theta \varphi, \theta} = 0, \tag{2}
\]

\[
\Gamma^\varphi_{\varphi \varphi} = \frac{1}{2r^2 \sin^2 \theta} g^{\varphi \varphi, \varphi} = 0, \tag{6}
\]

\[
\Gamma^\varphi_{\theta \varphi} = \frac{1}{2r^2 \sin^2 \theta} g^{\varphi \varphi, \theta} = \cos \theta / \sin \theta, \tag{5}\]

\[
\Gamma^\varphi_{\varphi \varphi} = \frac{1}{2r^2 \sin^2 \theta} g^{\varphi \varphi, \varphi} = 0, \tag{4}\]

\[
\Gamma^\varphi_{\varphi \theta} = \frac{1}{2r^2 \sin^2 \theta} (2g^{\varphi \varphi, \theta} - g^{\theta \theta, \varphi} - g^{\theta \varphi, \theta}) = -\sin \theta \cos \theta. \tag{3}\]

(ii) Write the equations of parallel transport of a tangent vector \( \mathbf{W} = W^\theta \partial_\theta + W^\varphi \partial_\varphi \) along a small circle with constant \( \theta = \theta_0 \) – the first thing you will need is to do is find a vector \( \mathbf{V} \) tangent to this curve.

Suppose we parametrize the curve by some parameter \( \lambda \). The tangent vector \( d/d\lambda = \mathbf{V} = V^\theta \partial_\theta + V^\varphi \partial_\varphi \) has components \( V^\theta = d\theta/d\lambda \) and \( V^\varphi = d\varphi/d\lambda \). Since the curve has constant \( \theta \), we find that \( V^\theta = 0 \). Keeping the parametrization of the curve general for now, thus we have

\[
\mathbf{V} = \frac{d\varphi}{d\lambda} \partial_\varphi. \tag{7}\]

The equations of parallel transport are

\[
V^\mu \nabla_\mu W^\nu = 0 = \frac{dW^\nu}{d\lambda} + \Gamma^\nu_{\mu \sigma} V^\mu W^\sigma = \frac{dW^\nu}{d\lambda} + \Gamma^\nu_{\varphi \sigma} \frac{d\varphi}{d\lambda} W^\sigma. \tag{8}\]

We see that we can get rid of \( \lambda \) and write the parallel transport equation in terms of \( \varphi \) directly:

\[
\frac{dW^\nu}{d\varphi} = -\Gamma^\nu_{\varphi \sigma} W^\sigma. \tag{9}\]

We thus have two coupled equations:

\[
\frac{dW^\theta}{d\varphi} = -\Gamma^\theta_{\varphi \sigma} W^\sigma = \sin \theta_0 \cos \theta_0 W^\varphi, \tag{10}\]

\[
\frac{dW^\varphi}{d\varphi} = -\Gamma^\varphi_{\varphi \sigma} W^\sigma = -\cos \theta_0 W^\theta, \tag{11}\]

where I used the expressions for the Christoffel symbols derived above, evaluated on the curve of constant \( \theta = \theta_0 \).

(iii) Solve these equations, starting with some initial conditions \( W^\theta|_0 \) and \( W^\varphi|_0 \) at \( \varphi = 0 \). How does the vector compare to itself after being parallel-transported once around a small circle?

We can combine these equations into the second-order linear ODE:

\[
\frac{d^2W^\theta}{d\varphi^2} + \cos^2 \theta_0 W^\theta = 0, \tag{12}\]
Thus, after being transported once, we have

\[ W^\theta(\varphi) = W^\theta|_0 \cos[(\cos \theta_0)\varphi] + \frac{1}{\cos \theta_0} \frac{dW^\theta}{d\varphi} \bigg|_0 \sin[(\cos \theta_0)\varphi], \]  

(13)

\[ W^\varphi(\varphi) = W^\varphi|_0 \cos[(\cos \theta_0)\varphi] + \frac{1}{\cos \theta_0} \frac{dW^\varphi}{d\varphi} \bigg|_0 \sin[(\cos \theta_0)\varphi]. \]  

(14)

Now using the original equations, we can related the first derivative of \( W^\theta \) to \( W^\varphi \) and vice-versa. Thus we obtain, finally,

\[ W^\theta(\varphi) = W^\theta|_0 \cos[(\cos \theta_0)\varphi] + \sin \theta_0 \ W^\varphi|_0 \sin[(\cos \theta_0)\varphi], \]  

(15)

\[ W^\varphi(\varphi) = W^\varphi|_0 \cos[(\cos \theta_0)\varphi] - \frac{1}{\sin \theta_0} W^\theta|_0 \sin[(\cos \theta_0)\varphi]. \]  

(16)

Let us moreover define \( W^\hat{\varphi} = \sin \theta \ W^\varphi \). This is the component of \( W \) along \( e_\varphi = \frac{1}{\sin \theta} \partial_\varphi \), which has norm \( r \), like \( \partial_\theta \). Thus, after being transported once, we have

\[ \begin{pmatrix} W^\theta(2\pi) \\ W^\hat{\varphi}(2\pi) \end{pmatrix} = \begin{pmatrix} \cos(2\pi \cos \theta_0) & \sin(2\pi \cos \theta_0) \\ -\sin(2\pi \cos \theta_0) & \cos(2\pi \cos \theta_0) \end{pmatrix} \begin{pmatrix} W^\theta(0) \\ W^\hat{\varphi}(0) \end{pmatrix} \]  

(17)

We see that the vector \( W \) is rotated by an angle \( 2\pi \cos \theta_0 \) after being parallel-transported once around the small circle.

Let us consider a small circle near the pole, so \( \theta_0 \ll 1 \). We then have \( \cos \theta_0 \approx 1 - \theta_0^2/2 \), thus

\[ \cos(2\pi \cos \theta_0) - 1 \approx \cos(\pi \theta_0^2) - 1 = \mathcal{O}(\theta_0^4), \]  

(18)

\[ \sin(2\pi \cos \theta_0) \approx -\sin(\pi \theta_0^2) \approx -\pi \theta_0^2. \]  

(19)

We then get

\[ W^\theta(2\pi) - W^\theta(0) \approx -\pi \theta_0^2 \times W^\hat{\varphi}|_0, \]  

(20)

\[ W^\hat{\varphi}(2\pi) - W^\hat{\varphi}(0) \approx \pi \theta_0^2 \times W^\theta|_0. \]  

(21)

Therefore the difference is proportional to the area enclosed by the small circle.

**Exercise 2: Killing vector fields and conservation laws**

(i) Suppose that, in some coordinate system \( \{x^\mu\} \), the metric components do not depend on a specific coordinate \( x^\sigma \), i.e. \( \partial_\sigma g_{\mu\nu} = 0 \) for any \( \mu, \nu \). Show that this implies that \( P_\sigma^* \) (with the index downstairs!) is constant along geodesics with tangent vector \( P^\alpha \).

Suppose \( P = d/d/\lambda \), i.e. that \( \lambda \) is the parameter along the geodesic (proper time or affine parameter for null geodesics). Since \( P^\mu \nabla_\mu P^\nu = 0 \), and the covariant derivative is metric-compatible, we also automatically have \( P^\mu \nabla_\mu P_\nu = 0 \). Applying this to \( \nu = \sigma^* \), we have

\[ 0 = P^\mu \nabla_\mu P_{\sigma^*} = \frac{dP_{\sigma^*}}{d\lambda} - \Gamma^\delta_{\mu \sigma^*} P^\mu P_\delta. \]  

(22)

Let’s compute the relevant Christoffel symbol:

\[ \Gamma^\delta_{\mu \sigma^*} = \frac{1}{2} g^{\delta \rho} (g_{\rho \mu, \sigma^*} + g_{\sigma^* \rho, \mu} - g_{\sigma^* \mu, \rho}) = g^{\delta \rho} g_{\sigma^* [\rho, \mu]}, \]  

(23)

since \( g_{\rho \mu, \sigma^*} = 0 \). Thus we have

\[ \frac{dP_{\sigma^*}}{d\lambda} = g^{\delta \rho} g_{\sigma^* [\rho, \mu]} P^\mu P_\delta = g_{\sigma^* [\rho, \mu]} P^\mu P_\rho = 0, \]  

(24)

as the last term is the contraction of a pair of symmetric indices with a pair of antisymmetric indices. We thus showed that \( P_{\sigma^*} \) is constant along the geodesic. **Important:** \( P^\mu \nabla_\mu P_{\sigma^*} \) really means \( (\nabla P)^{\mu}_{\sigma^* \mu} \), and is not equal to \( dP_{\sigma^*}/d\lambda \), which is the quantity that must be zero for \( P_{\sigma^*} \) to be constant along the geodesic.
(ii) A Killing vector field $K^\alpha$ is a vector field that satisfies Killing’s equation, $\nabla_{(\alpha} K_{\beta)} = 0$. Show that, if $K^\alpha$ is a Killing vector field, then $K_\alpha P^\alpha$ is constant along geodesics with tangent vector $P^\alpha$.

For an arbitrary vector field $K^\alpha$, $P^\alpha K_\alpha$ is a scalar field, thus

$$\frac{d}{d\lambda} (K_\alpha P^\alpha) = P^\mu \nabla_\mu (K_\alpha P^\alpha) = P^\mu (\nabla_\mu K_\alpha) P^\alpha + P^\mu (\nabla_\mu P^\alpha) K_\alpha = (\nabla_\mu K_\alpha) P^\mu P^\alpha = (\nabla_\mu (K_\alpha)) P^\mu P^\alpha$$

(25)

since $P^\mu \nabla_\mu P^\alpha = 0$, and the tensor product $P^\mu P^\alpha$ is symmetric. Thus, if $K^\alpha$ is a Killing vector field, this vanishes, thus $K_\alpha P^\alpha$ is constant along geodesics.

(iii) Show that, given a conserved stress-energy tensor $T^{\alpha \beta}$, i.e. such that $\nabla_\mu T^{\alpha \beta} = 0$, and a Killing vector field $K^\alpha$, the current $J^\alpha \equiv K_\beta T^{\beta \alpha}$ is conserved (i.e. divergence-free).

$$\nabla_\alpha J^\alpha = \nabla_\alpha (K_\beta T^{\beta \alpha}) = (\nabla_\alpha K_\beta) T^{\beta \alpha} + K_\beta \nabla_\alpha T^{\beta \alpha} = (\nabla_\alpha (K_\beta)) T^{\beta \alpha},$$

(26)

since $T^{\alpha \beta}$ is divergence-free, and symmetric (so contractions with it only pick the symmetric part). Thus, if $K^\alpha$ is a Killing vector field, this vanishes.

(iv) In a coordinate system $\{x^\mu\}$, define $K = \partial_{\sigma^*}$, for some specific coordinate $x^{\sigma^*}$. Show that $K^\alpha$ satisfies Killing’s equation if and only if $\partial_{\sigma^*} g_{\mu \nu} = 0$ for all $\mu, \nu$.

Let us write explicitly the symmetric part of the gradient of $K_\alpha$:

$$\nabla_{(\alpha} K_{\beta)} = \partial_{(\alpha} K_{\beta)} - \Gamma^\gamma_{\alpha \beta} K_\gamma,$$

(27)

where the second term is automatically symmetric. Now since $K = \partial_{\sigma^*}$, we have

$$K^\mu = \delta^\mu_{\sigma^*}, \quad K_\beta = g_{\beta \mu} K^\mu = g_{\beta \sigma^*}.$$  

(28)

So we found

$$\nabla_{(\alpha} K_{\beta)} = \partial_{(\alpha} g_{\beta \sigma^*} - g_{\beta \sigma^*} \Gamma^\delta_{\alpha \beta} = \partial_{(\alpha} g_{\beta \sigma^*} - \frac{1}{2} (g_{\alpha \sigma^*, \beta} + g_{\beta \sigma^*, \alpha} - g_{\alpha \beta, \sigma^*}).$$

(29)

where we used the expression for the Christoffel symbol and simplified the contraction with the inverse metric. The first two terms in the parenthesis are exactly $\partial_{(\alpha} g_{\beta \sigma^*)}$. Thus we have

$$\nabla_{(\alpha} K_{\beta)} = \frac{1}{2} g_{\alpha \beta, \sigma^*}. $$

(30)

Hence we see that $K = \partial_{\sigma^*}$ is a Killing vector field (i.e. the left-hand-side vanishes for any pair $\alpha, \beta$, by definition) if and only if the metric components do not depend on $\sigma^*$.

The concept of Killing vectors is important and you should remember it!

Exercise 3: Ideal fluid

Preliminaries: A projector is a tensor $P^\alpha_\beta$ which satisfies $P^\alpha_\beta P^\beta_\gamma = P^\alpha_\gamma$. A projector parallel to a vector $X^\alpha$ is a projector $(P_\parallel)^\alpha_\beta$ satisfying $(P_\parallel)^\alpha_\beta Y^\beta \propto X^\alpha$ for any vector $Y^\alpha$. A projector perpendicular to a vector $X^\alpha$ is a projector $(P_\perp)^\alpha_\beta$ satisfying $X^\alpha P^\alpha_\beta Y^\beta = 0$, for any vector $Y^\alpha$.

(i) Suppose an ideal fluid has 4-velocity $u^\alpha$, normalized as usual such that $u^\alpha u_\alpha = -1$. Construct the projectors parallel to and perpendicular to the fluid’s 4-velocity. Rewrite the stress-energy tensor of the ideal fluid $T^{\alpha \beta} = \rho_\text{fl} u^\alpha u^\beta + P_\text{fl} (g^{\alpha \beta} + u^\alpha u^\beta)$ in terms of these projectors.

The desired projectors are

$$(P_\parallel)^\alpha_\beta = -u^\alpha u_\beta, \quad (P_\perp)^\alpha_\beta = \delta^\alpha_\beta + u^\alpha u_\beta.$$  

(31)
Let us first prove that they are indeed projectors:

\[(P_\parallel)^\alpha_\beta (P_\parallel)^\beta_\gamma \gamma = u^\alpha u_\beta u^\beta u_\gamma = -u^\alpha u_\gamma = (P_\parallel)^\alpha_\gamma,\]

since \(u_\mu u^\mu = -1\).

\[(P_\perp)^\alpha_\beta (P_\perp)^\beta_\gamma \gamma = (\delta_\alpha^\beta + u^\alpha u_\beta)(\delta_\gamma^\beta + u_\gamma u_\gamma) = \delta_\alpha^\gamma + 2u^\alpha u_\gamma + u^\alpha u_\beta u_\beta u_\gamma = \delta_\alpha^\gamma + u^\alpha u_\gamma = (P_\perp)^\alpha_\gamma.\]

It should be clear that \((P_\parallel)^\alpha_\beta Y^\beta = -(Y^\beta u_\beta)u^\alpha\) is along \(u^\alpha\), as it should. Moreover,

\[u_\alpha (P_\perp)^\alpha_\beta Y^\beta = u_\alpha (\delta_\alpha^\beta + u^\alpha u_\beta)Y^\beta = u_\beta Y^\beta (1 + u_\alpha u^\alpha) = 0.\]

Recalling that \(g_\beta^\alpha = \delta_\beta^\alpha\), we have

\[T^{\alpha\beta} = -\rho_{\text{tf}}(P_\parallel)^{\alpha\beta} + P_{\text{tf}} (P_\perp)^{\alpha\beta}.\]

\[(ii) \text{ What physical quantity does the scalar field } \dot{Q} \equiv -u_\alpha \nabla_\beta T^{\alpha\beta} \text{ represent? Show that } \dot{Q} = \nabla_\alpha (\rho_{\text{tf}} u^\alpha) + P_{\text{tf}} \nabla_\alpha u^\alpha.\]

In the rest-frame of the fluid, the components of the 4-velocity are, by definition \(u^\mu = (1, 0, 0, 0)\). Thus, taking inertial coordinates in that rest-frame, we have \(u_\mu = (-1, 0, 0, 0)\) in the fluid’s rest frame. Thus, \(\dot{Q} = +\nabla_\beta T^{0\beta}\) in the fluid’s rest-frame. This is the zero-th component of the 4-force density (see lecture 10). A 4-force is a rate of change of momentum. The zero-th component of 4-momentum is energy. Thus, \(\dot{Q}\) is the rate of heating per unit volume, in the fluid’s rest-frame – heating is just a rate of change of energy! In general, whenever we multiply a quantity by \(-u_\alpha\), it represents the value of the zero-th component of this quantity in the rest-frame of the “observer” whose 4-velocity is \(u^\alpha\).

Let us now compute this heating rate:

\[\dot{Q} = -u_\alpha \nabla_\beta T^{3\alpha} = -\nabla_\beta (u_\alpha T^{3\alpha}) + T^{3\alpha} \nabla_\beta u_\alpha.\]

The first term in parenthesis only picks the component of \(T^{3\alpha}\) parallel to \(u^\alpha\). In addition,

\[u^\alpha \nabla_\beta u_\alpha = \frac{1}{2} \nabla_\beta (u^\alpha u_\alpha) = \frac{1}{2} \nabla_\beta (-1) = 0.\]

Thus the second term only picks the components of \(T^{3\alpha}\) proportional to \(g^{\alpha\beta}\). Hence we find the desired expression.

\[(iii) \text{ Suppose that the ideal fluid is made of particles with conserved number. This means that there exists a number } 4\text{-current } N^\alpha, \text{ such that } N^0 \text{ is the number density and } \dot{N} \text{ is the number flux, which satisfies } \nabla_\alpha N^\alpha = 0. \text{ Denoting by } n_{\text{tf}} \text{ the particle number density in the fluid’s rest-frame, derive the expression for the particle number current } N^\alpha \text{ in an arbitrary frame}.\]

By definition, in the fluid’s rest frame, \(N^0 = n_{\text{tf}}\) and \(N^i = 0\), since this frame was defined as being isotropic, hence there is no preferred direction for the number flux. Now in the fluid’s rest frame we have \(u^0 = 1\) and \(u^i = 0\). So in that frame, \(N^\mu = n_{\text{tf}} u^\mu\). This is a vector equality, which should hold in any frame – note that \(n_{\text{tf}}\) is a scalar field, defined as the fluid number density in the fluid’s rest frame.

\[(iv) \text{ We define } \epsilon_{\text{tf}} \equiv \rho_{\text{tf}}/n_{\text{tf}}. \text{ What physical quantity does this represent? Combining particle number conservation with question } (ii), \text{ derive an equation involving } \dot{Q} \text{ and } \epsilon_{\text{tf}}, \text{ and explain why it represents the first law of thermodynamics}.\]

\(\epsilon_{\text{tf}}\) is the ratio of energy density in the fluid’s rest frame to the number density in the fluid’s rest frame. It is thus the mean energy per particle in the fluid’s rest frame.

\[\dot{Q} = \nabla_\alpha (\epsilon_{\text{tf}} n_{\text{tf}} u^\alpha) + P_{\text{tf}} \nabla_\alpha \left( \frac{1}{n_{\text{tf}}} n_{\text{tf}} u^\alpha \right) = \nabla_\alpha (\epsilon_{\text{tf}} N^\alpha) + P_{\text{tf}} \nabla_\alpha \left( \frac{N^\alpha}{n_{\text{tf}}} \right) = n_{\text{tf}} u^\alpha (\nabla_\alpha \epsilon_{\text{tf}} + P_{\text{tf}} \nabla_\alpha (1/n_{\text{tf}})),\]

where we used \(\nabla_\alpha N^\alpha = 0\) and substituted \(N^\alpha = n_{\text{tf}} u^\alpha\). Let us now consider a small volume \(V\) enclosing a fixed number of particles \(N_{\text{part}} = n_{\text{tf}} V\), where we work in the fluid’s rest frame. We may thus replace \(1/n_{\text{tf}} = V/N_{\text{part}}\). We also define the total energy of these particles by \(U = N_{\text{part}} \epsilon_{\text{tf}}\). Multiplying the above equation by \(V\), we get

\[V \dot{Q} = u^\alpha \nabla_\alpha U + P_{\text{tf}} u^\alpha \nabla_\alpha V = \frac{dU}{dt} + P_{\text{tf}} \frac{dV}{dt}\]

The left-hand side is the rate of heating of the volume \(V\). This is exactly the first law of thermodynamics: \(\delta Q = dU + PdV\), written in differential form.