General Relativity Fall 2019
Lecture 11: The Riemann tensor

Yacine Ali-Haïmoud
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The Riemann tensor quantifies the **curvature of spacetime**, as we will see in this lecture and the next.

RIEMANN TENSOR: BASIC PROPERTIES

Definition – Given any vector field V^α, $\nabla_{[\alpha}\nabla_{\beta]}V^\gamma$ is a tensor field. Let us compute its components in some coordinate system:

$$\begin{aligned}\nabla_{[\mu}\nabla_{\nu]}V^\sigma &= \partial_{[\mu}(\nabla_{\nu]}V^\sigma) - \Gamma^\lambda_{[\mu\nu]}\nabla_\lambda V^\sigma + \Gamma^\sigma_{\lambda[\mu}\nabla_{\nu]}V^\lambda \\ &= \partial_{[\mu}(\partial_{\nu]}V^\sigma + \Gamma^\sigma_{\nu]\lambda}V^\lambda) + \Gamma^\sigma_{\lambda[\mu}\left(\partial_{\nu]}V^\lambda + \Gamma^\lambda_{\nu]\rho}V^\rho\right) \\ &= \left(\partial_{[\mu}\Gamma^\sigma_{\nu]\lambda} + \Gamma^\sigma_{\rho[\mu}\Gamma^\rho_{\nu]\lambda}\right)V^\lambda \equiv \frac{1}{2}R^\sigma{}_{\lambda\mu\nu}V^\lambda, \end{aligned} \tag{1}$$

where all partial derivatives of V^μ cancel out after antisymmetrization.

Since the left-hand side is a tensor field and V is a vector field, we conclude that $R^\sigma{}_{\lambda\mu\nu}$ is a tensor field as well – this is the **tensor division theorem**, which I encourage you to think about on your own. You can also check that explicitly from the transformation law of Christoffel symbols.

This is the **Riemann tensor**, which **measures the non-commutation of second derivatives of vector fields** – remember that second derivatives of scalar fields do commute, by assumption. It is **completely determined by the metric**, and is **linear in its second derivatives**.

Expression in LICS – In a LICS the Christoffel symbols vanish **but not their derivatives**. Let us compute the latter:

$$\partial_\mu\Gamma^\sigma_{\nu\lambda} = \frac{1}{2}\partial_\mu\left[g^{\sigma\delta}\left(\partial_\nu g_{\lambda\delta} + \partial_\lambda g_{\nu\delta} - \partial_\delta g_{\nu\lambda}\right)\right] = \frac{1}{2}\eta^{\sigma\delta}\left(\partial_\mu\partial_\nu g_{\lambda\delta} + \partial_\mu\partial_\lambda g_{\nu\delta} - \partial_\mu\partial_\delta g_{\nu\lambda}\right), \tag{2}$$

since the first derivatives of the metric components (thus of its inverse as well) vanish in a LICS. Therefore we get, in a LICS,

$$R_{\delta\lambda\mu\nu} = \frac{1}{2}\left(\partial_\lambda\partial_\mu g_{\nu\delta} - \partial_\lambda\partial_\nu g_{\mu\delta} + \partial_\delta\partial_\nu g_{\mu\lambda} - \partial_\delta\partial_\mu g_{\nu\lambda}\right). \tag{3}$$

Symmetries – The Riemann tensor $R^\sigma{}_{\lambda\mu\nu}$ is, by definition, *(i)* **antisymmetric in its last two indices**. From the expression in the LICS, we further see that the fully covariant tensor $R_{\sigma\lambda\mu\nu}$ is moreover *(ii)* **antisymmetric in the first two indices**, *(iii)* **symmetric under exchange of the first and last pair**, and *(iv)*, satisfies the following identity:

$$R_{\sigma\lambda\mu\nu} + R_{\sigma\mu\nu\lambda} + R_{\sigma\nu\lambda\mu} = 0. \tag{4}$$

Although derived in a specific coordinate system, these symmetry properties are tensorial and **remain true in any coordinate system**.

Ricci tensor and Ricci scalar – We may define the Ricci tensor and scalar, respectively, as $\boxed{R_{\alpha\beta} \equiv R^\gamma{}_{\alpha\gamma\beta}}$ and $\boxed{R \equiv R^\alpha{}_\alpha}$ (it is customary to use the same letter for all...). The **Ricci tensor is symmetric**.

Number of independent components of Riemann – The number of independent components in each anti-symmetric pair of indices is $N = n(n-1)/2$. If only the first 3 symmetry conditions were satified, we would have $N(N+1)/2$ independent components. We now want to know how many **independent constraints** the fourth symmetry property provides. Define $R_{\delta\{\lambda\mu\nu\}} \equiv R_{\delta\lambda\mu\nu} + R_{\delta\mu\nu\lambda} + R_{\delta\nu\lambda\mu}$. If any two indices are the same, then this quantity automatically vanishes due to the antisymmetry of Riemann in its first two and last two indices (check it!). Therefore, the fourth symmetry property adds information only if all 4 indices are different. This means that, in dimension n, it provides $C_n^4 = n(n-1)(n-2)(n-3)/24$ independent conditions. Therefore, the **total number of independent components of Riemann** is

$$\frac{1}{2}\frac{n(n-1)}{2}\left(\frac{n(n-1)}{2}+1\right) - \frac{n(n-1)(n-2)(n-3)}{24} = \frac{n^2(n^2-1)}{12}. \tag{5}$$

This is precisely the difference between the number of second derivatives of the metric tensor $\partial_\delta \partial_\sigma g_{\mu\nu}$ and the number of third derivatives of coordinates $\partial^3 x^\delta / \partial x^\lambda \partial x^\mu \partial x^\nu$, i.e

$$\left(\frac{n(n+1)}{2}\right)^2 - n\frac{n(n+1)(n+2)}{6} = \frac{n^2(n^2-1)}{12}. \tag{6}$$

In other words, the independent components of the Riemann tensor can be thought of as the $n^2(n^2-1)/12$ **(linear combinations) of second derivatives of the metric tensor that cannot be set to zero by coordinate transformations**.

Curvature as a function of dimension, Weyl tensor

• In dimension $n = 1$, the Riemann tensor has **0 independent components**, i.e. vanishes everywhere. There is no **intrinsic curvature** in 1-dimension. An ant walking on a line does not feel curvature (even if the line has an extrinsic curvature if seen as embedded in $\mathbb{R}^2$).

• In dimension $n = 2$, the Riemann tensor has **1 independent component**. It is therefore entirely determined by the Ricci scalar, or scalar curvature: $R_{\alpha\beta\gamma\delta} = R\, g_{\alpha[\gamma}g_{\delta]\beta}$.

• In dimension $n = 3$, the Riemann tensor has **6 independent components**, just as many as the symmetric Ricci tensor. The Riemann tensor is entirely determined by the 6 independent components of the Ricci tensor:

$$R_{\alpha\beta\gamma\delta} = (g_{\alpha\gamma}R_{\beta\delta} - g_{\alpha\delta}R_{\beta\gamma} - g_{\beta\gamma}R_{\alpha\delta} + g_{\beta\delta}R_{\alpha\gamma}) + \frac{R}{2}(g_{\beta\gamma}g_{\alpha\delta} - g_{\alpha\gamma}g_{\beta\delta}). \tag{7}$$

One can check that this expression gives the Ricci tensor upon contraction.

• Finally, in $n \geq 4$, the Riemann tensor contains more information than there is in Ricci: we define the **trace-free Weyl tensor** $C_{\alpha\beta\gamma\delta}$ such that

$$R_{\alpha\beta\gamma\delta} = \frac{1}{n-2}(g_{\alpha\gamma}R_{\beta\delta} - g_{\alpha\delta}R_{\beta\gamma} - g_{\beta\gamma}R_{\alpha\delta} + g_{\beta\delta}R_{\alpha\gamma}) + \frac{R}{(n-1)(n-2)}(g_{\beta\gamma}g_{\alpha\delta} - g_{\alpha\gamma}g_{\beta\delta}) + C_{\alpha\beta\gamma\delta}. \tag{8}$$

The first two pieces have the correct symmetries, and, when contracted, give the Ricci tensor and scalar. The remainder $C_{\alpha\beta\gamma\delta}$ has the same symmetries as the Riemann tensor, and is in addition trace-free, $C^\alpha{}_{\beta\alpha\gamma} = 0$. This means that it has

$$\frac{n^2(n^2-1)}{12} - \frac{n(n+1)}{2} = \frac{n(n+1)}{2}\left[\frac{n(n-1)}{6} - 1\right] \tag{9}$$

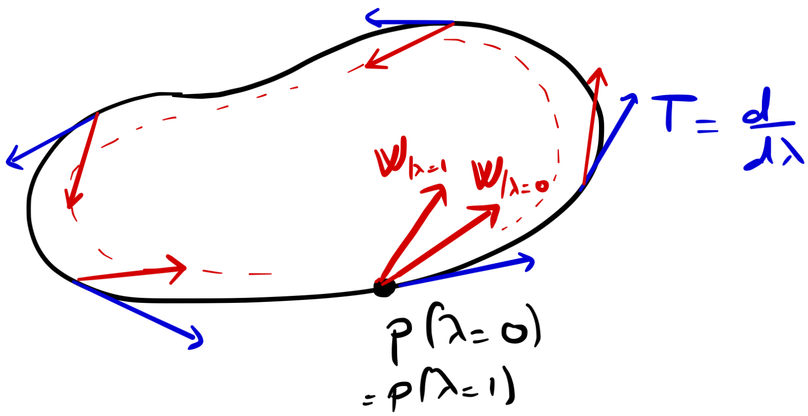
independent components. **In dimension 4, the Riemann tensor has 20 independent components, 10 of which are in the Weyl tensor.**

Note: To prove these equations, first prove them in a LICS, where they are easy to derive; they must then hold in an arbitrary basis, as they are tensorial. In a tensorial form, the above equations take the form

$$\textbf{Riemann} \sim \boldsymbol{g} \otimes \textbf{Ricci} + R\, \boldsymbol{g} \otimes \boldsymbol{g} + \boldsymbol{C}. \tag{10}$$

PARALLEL TRANSPORT ALONG A SMALL CLOSED LOOP

Let us consider a vector V defined at a point p of the manifold, and a **small closed curve** passing through p, with tangent vector $T = d/d\lambda$. We define the vector field W on the curve by parallel-transporting V, i.e. such that $W|_p = V$, and $\nabla_T W = 0$. We then ask what is W at p after being parallel-transported once around the curve.



By assumption, we have

$$0 = T^\nu \nabla_\nu W^\mu = T^\nu \partial_\nu W^\mu + T^\nu \Gamma^\mu{}_{\nu\sigma} W^\sigma = \frac{dW^\mu}{d\lambda} + \Gamma^\mu{}_{\nu\sigma}\frac{dx^\nu}{d\lambda} W^\sigma. \tag{11}$$

Let us pick an **LICS** $\{x^\mu\}$ **centered at** p (so p has coordinates (0, 0, 0, 0) in this system). The Christoffel symbols vanish at p, but **only** there: elsewhere, they are small, but non-zero. To lowest order, $\Gamma^\mu{}_{\nu\sigma} \approx x^\gamma \partial_\gamma \Gamma^\mu{}_{\nu\sigma}|_p$, and $W^\sigma \approx V^\sigma$ plus a small correction, which we can compute by substituting $W^\sigma \approx V^\sigma$ in the right-hand-side. We therefore have

$$W^\mu - V^\mu \approx -\partial_\gamma \Gamma^\mu{}_{\nu\sigma}|_p V^\sigma \int d\lambda \frac{dx^\nu}{d\lambda} x^\gamma. \tag{12}$$

The last integral is antisymmetric: indeed, its symmetric part is

$$\int d\lambda \left(\frac{dx^\nu}{d\lambda}x^\gamma + \frac{dx^\gamma}{d\lambda}x^\nu\right) = \int d\lambda \frac{d}{d\lambda}(x^\gamma x^\nu) = x^\gamma x^\nu|_p - x^\gamma x^\nu|_p = 0, \tag{13}$$

since we are integrating over a **closed loop**. Therefore, we find

$$W^\mu - V^\mu \approx \partial_{[\nu}\Gamma^\mu_{\gamma]\sigma} V^\sigma \int dx^{[\nu} x^{\gamma]} = \frac{1}{2} R^\mu{}_{\sigma\nu\gamma} V^\sigma \int dx^{[\nu} x^{\gamma]}, \tag{14}$$

where we used the expression for the Riemann tensor in a LICS (i.e. setting the Christoffel symbols to zero, but not their derivatives). The last integral is just the **area enclosed by the curve**. Take for example a circle in the x^1, x^2 coordinates, parameterized by $x^1 = r\cos\lambda$, $x^2 = r\sin\lambda$, with $\lambda \in [0, 2\pi]$. Then $x^{[1}dx^{2]} = \frac{1}{2}r^2 d\lambda$, and $\int x^{[1}dx^{2]} = \pi r^2$. You will work out examples in the homework. From a dimensional analysis, the **Riemann tensor has dimensions of inverse length squared**. The corresponding characteristic lengthscale can be seen as the radius of curvature of spacetime. While you need not remember the exact prefactors (but should be able to re-derive them quickly!), do remember that

$$\boxed{\Delta V \sim V \times \text{Riemann} \times \text{Area of loop}}. \tag{15}$$