RIEMANN = 0 ⇔ SPACETIME IS FLAT

If spacetime is flat, there exists a globally inertial coordinate system, such that \( ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu \). In these coordinates, the second derivatives of the metric vanish, and the Riemann tensor vanishes identically. This proves the \( \Leftrightarrow \) implication. Let us now prove the \( \Rightarrow \).

We saw that the change of a vector upon parallel transport around a small closed loop is proportional to the Riemann tensor times the area of the loop. Equivalently, we could have shown that the difference in a vector parallel transported along two different paths is proportional to the Riemann tensor times the area of the loop enclosed by the two paths.

Therefore, if the Riemann tensor vanishes everywhere, parallel transport is independent of the path (assuming the manifold is simply connected). This holds even for a finite path. To prove this, subdivide a finite loop of area \( A \) into \( N \) loops of area \( (A/N) \). If Riemann vanishes, then the difference in parallel transport per loop (for small enough loop) scales as \( (A/N) \), where the index \( \alpha > 1 \). Hence the total error on a finite path scales as \( N(A/N)^\alpha \propto A^\alpha /N^{\alpha-1} \). This goes to zero as \( N \to \infty \).

Suppose the Riemann vanishes everywhere. At a given point \( p \in \mathcal{M} \), it is always possible to find an orthonormal basis of the tangent space, \( \{e_{(\mu)}\} \). The fact that the basis is orthonormal means that \( g(e_{(\mu)}, e_{(\nu)}) = \eta_{\mu\nu} \) at \( p \). Now parallel transport the \( e_{(\mu)} \) everywhere in the manifold. This is a well-defined operation, as parallel transport is path-independent because the Riemann tensor vanishes. So, for any vector field \( V^\alpha \), we have \( V^\alpha \nabla_\alpha e_{(\mu)} = 0 \). This implies that \( \nabla e_{(\mu)} = 0 \). Since \( \nabla \) is metric compatible, the \( (n(n+1)/2 \) scalar fields \( g(e_{(\mu)}, e_{(\nu)}) \) have zero gradient everywhere, i.e. are constant and equal to \( \eta_{\mu\nu} \), their value at the initial point \( p \). But those are just the components of \( g \). This means that the vector fields \( \{e_{(\mu)}\} \) form an orthonormal basis at any point of the manifold (not just the point \( p \) where they were initially defined). We denote by \( \{e^{(\mu)}\} \) the dual basis. Taking the gradient of \( e^{(\mu)}, e_{(\nu)} = \delta^\mu_\nu \), we find that, for any \( \mu, \nu \), \( (\nabla e^{(\mu)}), e_{(\nu)} = 0 \). Thus the dual vectors also have zero gradient, \( \nabla e^{(\mu)} = 0 \).

We are not done yet: we want to find globally inertial coordinates \( \{x^\mu\} \) such that \( ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu \). Since we have already obtained \( ds^2 = \eta_{\mu\nu} e^{(\mu)} \otimes e^{(\nu)} \), let us thus construct coordinates such that \( dx^\mu = e^{(\mu)} \). Suppose we start from some arbitrary coordinates \( \{x^{\mu'}\} \), with associated differential forms \( dx^{\mu'} \), which form a basis of the dual vector space. We may decompose \( e^{(\mu)} \) on this basis: \( e^{(\mu)} = e^{(\mu')} dx^{\mu'} \), i.e. \( e^{(\mu')} \) is the \( \mu' \)-th component of \( e^{(\mu)} \) on the basis \( \{dx^{\mu'}\} \). On the other hand, we have \( dx^\mu = (\partial x^\mu / \partial x^{\mu'}) dx^{\mu'} \). Thus we want to find coordinates \( x^\mu \) that solve the following partial differential equations:

\[
\frac{\partial x^\mu}{\partial x^{\mu'}} = e^{(\mu')}.
\]

(1)

Taking one more derivative with respect to \( x^{\mu'} \), and invoking the fact that partial derivatives in the same coordinate system commute, we find that this implies that the components \( e^{(\mu')}_{\mu'} \) must satisfy

\[
\partial_{\nu'} e^{(\mu')} = \partial_{\mu'} e^{(\nu')} \Rightarrow \partial_{\nu'} e^{(\mu')} = 0.
\]

(2)

This is a necessary condition for the partial differential equations (1) to have a solution. It turns out that it is also a sufficient condition. Now, we know that \( \nabla_\alpha e^{(\mu')} = 0 \), thus \( \nabla_\alpha e^{(\mu')} = 0 \). By the symmetry of the Christoffel symbols, this implies \( \partial_{\mu'} e^{(\mu')} = 0 \) for any coordinates \( \{x^{\mu'}\} \). Thus the condition is indeed satisfied, and we can indeed solve for coordinates \( \{x^{\mu}\} \) such that \( dx^\mu = e^{(\mu)} \), which are therefore a globally inertial coordinate system. Note that even if the coordinates \( \{x^{\mu}\} \) only cover a subset of the manifold, we may solve \( \partial \xi^\mu / \partial x^{\mu'} = e^{(\mu')}_{\mu'} \) for \( x^\mu \) as a function of \( x^{\mu'} \) locally, then switch to other coordinates \( x^{\mu''} \), etc.

To conclude, if Riemann = 0, we can find global coordinates \( \{x^\mu\} \) in which \( g = \eta_{\mu\nu} dx^\mu dx^\nu \), i.e. the vanishing of Riemann implies that spacetime is flat.
FERMI NORMAL COORDINATES

Consider a fiducial timelike geodesic $G$ (i.e. $G$ is a worldline on the manifold) in a curved manifold $\mathcal{M}$. We will build a coordinate system $(t, x^i)$ defined in a neighborhood of $G$, such that the geodesic is at zero spatial coordinates $x^i|_G = 0$, the metric is Minkowski along the geodesic, $g_{\mu\nu}|_G = \eta_{\mu\nu}$, and the first derivatives of the metric vanish along the geodesic, i.e. $\partial_\lambda g_{\mu\nu}|_G = 0$, hence that $\Gamma^\nu_{\mu\sigma}|_G = 0$. Note that this is more restrictive than a LICS: the metric is close to Minkowski not only at one event, but all along a curve! We will follow the proof of Manasse & Misner 1963.

On the geodesic, we define $x^0 = \tau$, the proper time, which we initialize at zero at some point $p_0 \in G$. At that “initial” point, we put the metric in normal form, i.e. define three spacelike vectors $e_0, e_1, e_2$ parallel-transporting them $G$-how they are constructed, by launching spacelike geodesics starting on $G$, with initial tangent vector $T = x^i e_i(0)$. This defines the coordinates $\{x^\mu\}$ in the vicinity of the geodesic, as follows. Given four small numbers $\{x^\mu\}$, we define the point $p(x^0, x^i)$ as follows. We build the spacelike geodesic starting on $G$ at time $\tau = x^0$, with tangent vector $T = x^i e_i(0)$ at $p(x^0, 0) \in G$. We parametrize the geodesic by $\lambda$, and define the point $p(x^\mu)$ to be the point reached when $\lambda = 1$. Explicitly, in some arbitrary coordinate system $\{x^{\mu}\}$, we solve for the following differential equations for the functions $x^{\mu'}(\lambda; x^0, x^i)$:

$$\frac{d^2 x^{\mu'}}{d\lambda^2} + \Gamma^{\mu'}_{\nu\sigma} \frac{dx^{\nu'}}{d\lambda} \frac{dx^{\sigma'}}{d\lambda} = 0,$$

(3)

$$x^{\mu'}|_{\lambda=0} = x^{\mu'}(p(x^0, 0)), \quad [p(x^0, 0) \in G],$$

(4)

$$\frac{dx^{\mu'}}{d\lambda}|_{\lambda=0} = x^i e^{\mu'}_i(0),$$

(5)

where $e^{\mu'}_i$ is the $\mu'$-th component of $e_i(\lambda)$ on the coordinate basis $\{\partial_{\mu(\lambda)}\}$. This defines the 4 functions $x^{\mu'}(\lambda; x^0, x^i)$, i.e. $x^{\mu'}(\lambda)$ given the parameters $x^0, x^i$. We then define $p(x^0, x^i)$ whose coordinates are $x^{\mu'}(\lambda = 1; x^0, x^i)$. This procedure does not necessarily work for arbitrary $x^i$: the geodesics we build do not necessarily extend to $\lambda = 1$. However, for small enough $x^i$, the geodesics are indeed well defined up to $\lambda = 1$, which is a point close to $G$. This defines the coordinates $\{x^\mu\}$, i.e. the mapping from an open set of $\mathbb{R}^4$ to $\mathcal{M}$.
Define $\tilde{\lambda} = \lambda/s$, where $s$ is a constant, thus $d/d\tilde{\lambda} = s\, d/d\lambda$. Thus we find that the above equations take exactly the same form for $(\lambda, x^i)$ and $(\tilde{\lambda}, sx^i)$. Hence, $x^\mu(\lambda; x^0, x^i) = x^\mu(\lambda/s; x^0, sx^i)$. In particular, setting $s = \lambda$, we find

$$x^\mu(\lambda; x^0, x^i) = x^\mu(1; x^0, \lambda x^i).$$

(6)

The right-hand-side are the $\{x^\mu\}$ coordinates of the point $p(x^0, \lambda x^i)$, by construction. Let us differentiate this equation with respect to $\lambda$, and evaluate it at $\lambda = 0$:

$$\frac{dx^\mu}{d\lambda}\bigg|_{\lambda=0} = x^i \frac{\partial x^\mu}{\partial x^i}.$$  

(7)

Note that $\partial/\partial x^i$ means “partial derivative with respect to the $i$-th variable”. From Eq. (5), we find that, for any $x^i$,

$$x^i \frac{\partial x^\mu}{\partial x^i} = x^i e^\mu_{(i)}.$$  

(8)

This implies

$$\frac{\partial x^\mu}{\partial x^i} |_{G} = e^\mu_{(i)}.$$  

(9)

On the other hand, $\partial^\mu_{(i)} = \frac{\partial x^\mu}{\partial x^i}$. Thus we conclude that

$$\frac{\partial(\mu)}{\partial x^i} |_{G} = e^{(i)}_{(i)}.$$  

(10)

since we already known that it holds for $\mu = 0$. Thus, the components of $g$ on the basis $\{x^\mu\}$ are indeed $g_{\mu\nu} |_{G} = g |_{G}(\partial_{(\mu)}, \partial_{(\nu)}) = \eta_{\mu\nu}$.

Let us now show that the Christoffel symbols vanish along $G$. Take one more derivative of Eq. (6) with respect to $\lambda$, and evaluate at $\lambda = 0$:

$$\frac{d^2x^\mu}{d\lambda^2}\bigg|_{\lambda=0} = x^i x^j \frac{\partial^2 x^\mu}{\partial x^i \partial x^j}.$$  

(11)

From Eqs. (3) and (5), we thus find, for any $x^i, x^j$,

$$\Gamma^\mu_{\nu\sigma}, x^i e^\nu_{(i)} x^j e^\sigma_{(j)} = x^i x^j \frac{\partial^2 x^\mu}{\partial x^i \partial x^j},$$  

(12)

hence, along the geodesic,

$$\Gamma^\mu_{\nu\sigma}, e^\nu_{(i)} e^\sigma_{(j)} = \frac{\partial^2 x^\mu}{\partial x^i \partial x^j}.$$  

(13)

This expression holds for any coordinates, in particular, it holds in the coordinate system $\{x^\mu\}$, for which $e^\mu_{(i)} = \partial^\mu_{(i)} = \delta^\mu_i$, and for which the right-hand-side is zero. Thus, we find, in the coordinates $\{x^\mu\}$, that

$$\Gamma^\mu_{ij} |_{G} = 0.$$  

(14)

Now recall that the four vectors $\{e_{(\mu)}\}$ are parallel-transported along the geodesic, whose tangent vector is $e_{(0)} = d/d\tau$. Thus, we have, in any coordinate system,

$$\frac{d^2 e^\mu_{(\mu)}}{d\tau^2} + e^\rho_{(0)} \Gamma^\mu_{\rho\sigma} e^\sigma_{(\mu)} = 0.$$  

(15)

In particular, in the $\{x^\mu\}$ coordinates, $e^\mu_{(\mu)} = \delta^\mu_i$ is constant and $e^\rho_{(0)} = \delta_0^\rho$, thus we find

$$\Gamma^\mu_{0\mu} |_{G} = 0.$$  

(16)

We have thus proved that the coordinates $\{x^\mu\}$ for inertial coordinates in the vicinity of the geodesic, all along the geodesic.

**Fermi-normal coordinates are the inertial coordinates that a free-falling observer would set in a neighborhood around them**, not just at one instant in time (this would be a LICS centered around that event), but all along their worldline.