• Keeping it real

Any measurable quantity in physics is always real. Think of complex numbers as a very useful intermediate step. They have rich mathematical properties, which we will explore.

• Refresher

We define $i$ such that $i^2 = -1$.

A general complex number $z = x + iy$, $x, y$ real numbers

$x = \text{Re}(z) =$ Real part

$y = \text{Im}(z) =$ Imaginary part

$\Rightarrow z = \text{Re}(z) + i \text{Im}(z)$

We can plot complex numbers on the complex plane

![Complex Plane Diagram](image)

- The complex conjugate of $z = x + iy$ is $z^* = x - iy$
  
  i.e. $\text{Re}(z^*) = \text{Re}(z)$, $\text{Im}(z^*) = -\text{Im}(z)$.

![Complex Conjugate Diagram](image)

- The magnitude or modulus of a complex number $z$ is

$$|z| = \sqrt{x^2 + y^2} = \sqrt{\text{Re}(z)^2 + \text{Im}(z)^2}$$

$zz^* = (x + iy)(x - iy) = x^2 + ixy - ixy - (iy)^2 = x^2 + y^2 = |z|^2$
The real and imaginary parts can be obtained from:

\[ \text{Re}(z) = \frac{z + z^*}{2}, \quad \text{Im}(z) = \frac{z - z^*}{2i} \]

**Complex exponential**

Remember: \( f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \) (Taylor expansion around 0)

Take \( f(x) = e^x \) \( \Rightarrow \frac{d^n f}{dx^n} \bigg|_{x=0} = 1 \)

\[ \Rightarrow e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \]

Generalize to any complex \( \# \):

\[ e^{\#} = \sum_{n=0}^{\infty} \frac{(\#)^n}{n!} \]

Let us apply this to \( z = i\theta \), \( \theta \) real number:

\[ \begin{align*}
  i^{2p} &= (i^2)^p = (-1)^p \\
  i^{2p+1} &= i \cdot i^{2p} = (-1)^p i
\end{align*} \]

\[ e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} = \sum_{\text{n even}} + \sum_{\text{n odd}} \]

\[ = \sum_{p=0}^{\infty} \frac{(i\theta)^{2p}}{(2p)!} + i \sum_{p=0}^{\infty} \frac{(i\theta)^{2p+1}}{(2p+1)!} = \cos \theta + i \sin \theta. \]

**Proof**

\[ f(x) = \cos x \]

\[ \frac{df}{dx} = -\sin x \quad \frac{d^2 f}{dx^2} = -\cos x \]

\[ \frac{d^3 f}{dx^3} = +\sin x \quad \frac{d^4 f}{dx^4} = +\cos x \quad \text{etc...} \]

\[ \Rightarrow \frac{d^{2p} f}{dx^{2p}} = (-1)^p \cos x \quad \frac{d^{2p+1} f}{dx^{2p+1}} = (-1)^{p+1} \sin x \quad \Rightarrow \frac{d^{2p} f}{dx^{2p}} \bigg|_0 = (-1)^p, \quad \frac{d^{2p+1} f}{dx^{2p+1}} \bigg|_0 = 0 \]
The Taylor expansion of $\cos \theta$ around $\theta = 0$ is

$$\cos \theta = \sum_{n=0}^{\infty} \frac{(-\theta)^n}{n!} = \sum_{n=2p}^{\infty} \frac{(-\theta)^{2p}}{(2p)!}$$

Similarly for the sine function:

$$\sin \theta = \sum_{p=0}^{\infty} \frac{(-\theta)^{2p+1}}{(2p+1)!}$$

$$\Rightarrow e^{i\theta} = \cos \theta + i \sin \theta$$

$$|e^{i\theta}| = (\cos^2 \theta + \sin^2 \theta)^{\frac{1}{2}} = 1$$

$$(e^{i\theta})^* = \cos \theta - i \sin \theta = e^{-i\theta}$$

$$\Rightarrow \cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \quad \sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

**Exercise:** Use this to derive $\cos^2 \theta = \frac{1 + \cos(2\theta)}{2}$

$\sin^2 \theta = \frac{1 - \cos(2\theta)}{2}$ etc...

Numbers of the form $e^{i\theta}$ lie on the circle of unit radius.
**Polar representation of complex numbers**

\[ z = r e^{i\theta} \]

\[ r = |z|, \quad \theta = \arg(z): \text{argument of } z. \]

\[ = r \cos \theta + i r \sin \theta \quad \Rightarrow \quad \theta = \arctan \left( \frac{\text{Im}(z)}{\text{Re}(z)} \right) \]

**Complex functions**

The set of complex numbers is denoted by \( \mathbb{C} \) (like \( \mathbb{R} \) for reals).

We now study functions \( f: \mathbb{C} \rightarrow \mathbb{C} \).

We can think of \( f \) as a function of \((x,y) = (\text{Re}(z), \text{Im}(z))\), which returns two real numbers:

\[ u(x,y) = \text{Re}(f(x,y)), \quad v(x,y) = \text{Im}(f(x,y)) \]

**Example.**

\[ f(z) = z^2 = (x+iy)^2 = x^2 + 2ixy + (iy)^2 = x^2 - y^2 + 2i xy \]

\[ = u(x,y) + i \ v(x,y) \quad \Rightarrow \quad u(x,y) = x^2 - y^2, \quad v(x,y) = 2xy. \]

**Derivatives of complex functions**

Given \( f: \mathbb{C} \rightarrow \mathbb{C} \), define

\[ \frac{df}{dz} \equiv \lim_{D \to 0} \frac{f(z+D) - f(z)}{D}, \quad (D \in \mathbb{C}). \]

The derivative \( \frac{df}{dz} \) exists if this limit is independent of the direction from which \( D \) approaches zero.
Write \( D = \zeta + i\eta \), with \( \zeta, \eta \) small real numbers.

\[
D \left( f(z + D) - f(z) \right) = \frac{D}{D} \left( f(x + \zeta, y + \eta) - f(x, y) \right) = \frac{\alpha \frac{\partial f}{\partial x} + \beta \frac{\partial f}{\partial y} + \cdots}{\alpha + i\beta}
\]

This must approach the same limit regardless of the ratio of \( \beta/\alpha \) as both \( \alpha, \beta \) approach 0.

Suppose \( \alpha \to 0, \beta \to 0 \) and \( \frac{\beta}{\alpha} \to 0 \)

(D gets "more and more real" as \( D \to 0 \)).

\[
\frac{\alpha \frac{\partial f}{\partial x} + \beta \frac{\partial f}{\partial y}}{\alpha + i\beta} = \frac{\frac{\partial f}{\partial x} + \frac{\beta}{\alpha} \frac{\partial f}{\partial y}}{1 + i\frac{\beta}{\alpha}} \to \frac{\partial f}{\partial x}
\]

Suppose \( \alpha \to 0, \beta \to 0 \) and \( \frac{\alpha}{\beta} \to 0 \)

(D gets "more and more imaginary" as \( D \to 0 \)).

\[
\frac{\alpha \frac{\partial f}{\partial x} + \beta \frac{\partial f}{\partial y}}{\alpha + i\beta} = \frac{\frac{\partial f}{\partial x} + \frac{\alpha}{\beta} \frac{\partial f}{\partial y}}{\frac{\alpha}{\beta} + i} \to \frac{i}{i} \frac{\partial f}{\partial y} = -i \frac{\partial f}{\partial y}
\]

For these two numbers to be equal, we need

\[
\Rightarrow \frac{\partial f}{\partial y} = i \frac{\partial f}{\partial x}
\]

\( \frac{\partial f}{\partial y} = i \frac{\partial f}{\partial x} \)

For these two numbers to be equal, we need

\[
\Rightarrow \frac{\partial f}{\partial y} = i \frac{\partial f}{\partial x}
\]

In terms of \((u, v) = (\text{Re}(f), \text{Im}(f))\), this implies:

\[
\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}, \quad \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}
\]

Causby-Riemann conditions
Def: A complex function $f : \mathbb{C} \to \mathbb{C}$ is **analytic** on some set $S \subseteq \mathbb{C}$ if

- the partial derivatives $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ exist on $S$,
- they satisfy the Cauchy-Riemann conditions.

**Exercise:**
- Show that if $f$ and $g$ satisfy the Cauchy-Riemann conditions, then $fg$ also does.
- Show that if $f$ is analytic, then $\frac{1}{f}$ is analytic everywhere except at $z_0$ s.t. $f(z_0) = 0$.

**Example:**

\[ f(z) = z^2 \quad u(x, y) = x^2 - y^2 \quad v(x, y) = 2xy \]

\[ \frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y} \quad \frac{\partial u}{\partial y} = -2y = -\frac{\partial v}{\partial x} \quad \checkmark \text{Analytic!} \]

\[ \frac{df}{dz} = \frac{df}{dx} = \frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} = 2x + 2iy = 2(x + iy) = 2z. \quad \text{As expected!} \]

**Exercise:** Show that $f(z) = z^n$ is analytic and prove that $\frac{df}{dz} = nz^{n-1}$.

**Counter-example:**

\[ f(z) = z \pm \frac{1}{z} = |z|^2 = x^2 + y^2 \]

\[ \Rightarrow u(x, y) = x^2 + y^2, \quad v(x, y) = 0 \]

\[ \frac{\partial u}{\partial x} = 2x \neq \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = 2y = -\frac{\partial v}{\partial x} \quad \text{except at a single point, } \pm \text{.} \]

\[ \Rightarrow f(z) = |z|^2 \text{ is not analytic.} \]

To understand why, consider $\Delta = e^{i\Theta} \in \mathbb{C}$

\[ f(z + \Delta) = |z + e^{i\Theta}|^2 = (z + e^{i\Theta})(z^* + e^{-i\Theta}) \]

\[ = z^2 + e^{i\Theta}z^* + e^{-i\Theta} z + e^{i\Theta} \]

\[ = f(z) + \epsilon (2e^{-i\Theta} + z^* e^{i\Theta}) + \epsilon^2 = f(z) + \Delta(z^* + e^{-2i\Theta} - 2) + \epsilon^2 \]
So \( f(z+\Delta) - f(z) \)
\[ \frac{D}{\Delta} \]

Now take \( \Delta \to 0 \), at constant \( \Theta \)
\( f(z+\Delta) - f(z) \to z^* + e^{-2i\Theta}z + \in e^{-i\Theta} \).

This result depends on the angle \( \Theta \) from which \( \Delta \) approaches 0.

\( \Rightarrow \) The derivative \( \frac{df}{dz} \) is not uniquely defined.

---

**Cauchy–Riemann integral theorem**

Consider an analytic function \( f \). For any closed curve \( \gamma \),
\[ \oint_{\gamma} f(z) \, dz = 0 \]

*What is the meaning of \( \oint_{\gamma} f(z) \, dz \)?*

\( \Rightarrow \) Chop \( \gamma \) into \( N \) segments of length \( \Delta z \), centered at \( z_n \).
\[ \oint_{\gamma} f(z) \, dz = \lim_{N \to \infty} \sum_{n=1}^{N} f(z_n) \Delta z. \]

\[ \oint_{\gamma} f(z) \, dz = \oint_{\gamma} (dx + idy) \left( u(x,y) + i v(x,y) \right) \]
\[ = \oint_{\gamma} (dx \, u(x,y) - dy \, v(x,y)) + i \oint_{\gamma} (dx \, v(x,y) + dy \, u(x,y)) \]

Define the vector field \( \vec{V} \) s.t.
\[ V_x = u(x,y), \quad V_y = -v(x,y), \quad V_z = 0. \]

\[ \oint_{\gamma} \vec{f} \cdot \vec{V} = \oint_{\gamma} (dx \, \hat{e}_x + dy \, \hat{e}_y) \cdot \vec{V} \]
\[ = \oint_{\gamma} (dx \, V_x + dy \, V_y) = \oint_{\gamma} (dx \, u(x,y) - dy \, v(x,y)) \]

Using Stokes' theorem, this is
\[ = \int_S d\vec{S} \cdot (\nabla \times \vec{V}) = \int_S dA \left( \frac{\nabla \times \vec{V}}{V} \right) \]
\[ = \int_S dA \left( \frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \right) = \int_S dA \left( -\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) = 0 \]
Similarly, using \( v_x = u(x,y) \) and \( v_y = u(x,y) + \text{Stokes' theorem, we find} \)
\[
\oint_{\gamma} (dx u(x,y) + dy u(x,y)) = \int_S \left( \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} \right) = 0.
\]

- Application: contour deformation

Example

Consider the integral \( I = \int_0^{+\infty} \sin(t^2) \)
\[
I = \lim_{T \to +\infty} \int_0^T dt \sin(t^2) \quad \text{(by definition)}
\]
Let us rewrite \( \sin(t^2) = \frac{e^{it^2} - e^{-it^2}}{2i} \) and define \( F(T) = \int_0^T e^{it^2} \, dt \).

In the homework, you'll show that if \( f \) and \( g \) are analytic on \( C \),
\( f \circ g : z \mapsto f(g(z)) \) is also analytic.
\( g(z) = e^{iz} \) is analytic, so \( \oint_{\gamma} \, dz \, e^{iz} = 0 \) for any closed curve.

Consider the closed contour \( \gamma \) made of
- \( \Gamma_1 : z = x \in [0, T] \)
- \( \Gamma_2 : z = Te^{i\theta}, \quad \theta \in [0, \pi/4] \)
- \( \Gamma_3 : z = Re^{it\pi/4}, \quad R \in [0, T] \).

\[
0 = \oint_{\gamma} \, dz \, e^{iz} = \int_0^T dx \, e^{ix^2} + \int_0^{\pi/4} Te^{i\theta} i\, d\theta \left( e^{iTe^{i\theta}} \right)^2 - \Gamma_2
\]

\[
d(Re^{it\pi/4}) \leftarrow \Gamma_3 \leftarrow \Gamma_2 \quad \text{with} \quad T = \text{constant}.
\]

\[
d(Re^{it\pi/4}) \quad \text{and} \quad \int_0^T dr \left( Re^{iT\pi/4} \right) = -\int_0^T dr...
\]
\[ F(t) = \int_0^T dx \, e^{ix} = -i T \int_0^{\pi/2} d\theta \, e^{i\theta} \, e^{iT e^{2i\theta}} + e^{i \pi} \int_0^T dR \, e^{i R^2 e^{i \pi/2}} = e^{-R^2} \]

Now \( e^{iT^2 e^{2i\theta}} = e^{iT^2 (\cos 2\theta + i \sin 2\theta)} = e^{iT^2 \cos 2\theta} e^{-T^2 \sin 2\theta} \)

* \( e^{iT^2 \cos 2\theta} \) has unit magnitude (e.g., Real Numbers) but unit magnitude

* For \( 0 < \theta < \frac{\pi}{4} \), \( \sin 2\theta > 0 \)

\[ \Rightarrow \text{as } T \to +\infty \text{, } e^{-\sin 2\theta T^2} \text{ goes to zero exponentially fast} \]

(Except for \( \Theta \geq \frac{1}{T^2} \) since \( \sin 2\Theta \approx 2\Theta \), but this is a smaller and smaller contribution to \( \int_0^{\pi/4} d\theta \), as \( T \to +\infty \)).

\[ \Rightarrow i T \int_0^{\pi/4} d\theta \, e^{i\theta} \, e^{iT e^{2i\theta}} = \int_{E_2} d\xi \, e^{i \xi^2} \to 0 \text{ as } T \to +\infty \]

\[ \Rightarrow \int_0^T dx \, e^{ix^2} \to e^{i \pi/4} \int_0^{+\infty} dR \, e^{-R^2} = e^{i \pi/4} \frac{\sqrt{\pi}}{2} \]

\[ \int_0^T dx \, \sin(x^2) = \text{Im} \left( \int_0^T dx \, e^{ix^2} \right) \to \text{Im} \left( e^{i \pi/4} \frac{\sqrt{\pi}}{2} \right) = \frac{\sqrt{\pi}}{2} \sin \frac{\pi}{4} \]

Conclusion: we found \( \int_0^{+\infty} dx \, \sin(x^2) = \frac{\sqrt{\pi}}{2\sqrt{2}} \).

This example illustrates one application of the Cauchy-Riemann Theorem: rewrite an apparently poorly defined integral in terms of a better-beloved one. This is particularly useful to compute integrals of sinusoidal functions, even numerically: it is always difficult to evaluate rapidly oscillating functions numerically.
Consider a function $f$ analytic on $C$, and $z_0 \in C$. The function $g(z) = \frac{f(z)}{z - z_0}$ is analytic everywhere except at $z = z_0$, where it is not defined (unless $f(z_0) = 0$).

For any contour $E$ that does not contain $z_0$, we can go through the same proof of Cauchy-Riemann theorem, and show that

$$\oint_E \frac{f(z)}{z - z_0} \, dz = 0.$$

However, if $E$ contains $z_0$, one cannot use Cauchy-Riemann: for $\oint_E f(z) \, dz$, this requires $g$ to be analytic everywhere inside $E$.

Repeating this procedure, we see that

$$\oint_E \frac{f(z)}{z - z_0} \, dz = \oint_{E_1} \frac{f(z)}{z - z_0} \, dz + \oint_{E_2} \frac{f(z)}{z - z_0} \, dz = 0.$$

For $z = z_0 + e^{i\theta}$, $0 \leq \theta \leq 2\pi$,

$$dz = e^{i\theta} \, d\theta$$

$$f(z) = f(z_0) + (z - z_0)f'(z_0) + \cdots$$

$$\Rightarrow \frac{f(z)}{z - z_0} = \frac{f(z_0)}{z - z_0} + \frac{f'(z_0)}{z - z_0} + \cdots = \frac{f(z_0)}{e^{i\theta}} + \frac{f'(z_0)}{e^{i\theta}} + \cdots$$

$$\Rightarrow \oint_E \frac{f(z)}{z - z_0} \, dz = \int_0^{2\pi} e^{i\theta} \, d\theta \left( \frac{f(z_0)}{e^{i\theta}} + \frac{f'(z_0)}{e^{i\theta}} + \cdots \right)$$

[Note: $\theta$ from $0$ to $2\pi$: $E$ needs to be oriented $\mathbf{C}$]
\[
\Rightarrow \oint_{E} \frac{f'(z)}{z-z_0} \, dz = \left[ \frac{f(z)}{z-z_0} \right]_{z=z_0} = \frac{f(z_0)}{z_0-z_0} = 0
\]

Differentiate this equation with respect to \( z_0 \):

\[
\frac{d}{dz_0} \left( \frac{1}{z-z_0} \right) = \frac{1}{(z-z_0)^2}
\]

\[
\Rightarrow \oint_{E} \frac{f'(z)}{(z-z_0)^2} \, dz = 2\pi i \frac{f''(z_0)}{2!} \quad (z_0 \text{ inside } E)
\]

Differentiate again:

\[
\Rightarrow 2 \oint_{E} \frac{f''(z)}{(z-z_0)^3} \, dz = 2\pi i \frac{d^2 f}{dz_0^2} \quad (z_0 \text{ inside } E)
\]

This generalizes to:

\[
n! \oint_{E} \frac{f^{(n)}(z)}{(z-z_0)^{n+1}} \, dz = 2\pi i \frac{d^n f}{dz_0^n} \quad (z_0 \text{ inside } E)
\]

**Implication:** If we know \( f(z) \) on some contour \( E \), then we know \( f(z) \) and all of its derivatives everywhere inside of \( E \) (at any \( z_0 \)).

---

*Complex Taylor series*

**Preliminaries:** \( |z_1 + z_2| \leq |z_1| + |z_2| \).

Now \( |z_1 + z_2|^2 = (z_1 + z_2)(z_1^* + z_2^*) = |z_1|^2 + |z_2|^2 + z_1 z_2^* + z_2 z_1^* \)

\[
= |z_1|^2 + |z_2|^2 + 2 \Re(z_1 z_2^*) \leq |z_1|^2 + |z_2|^2 + 2 |z_1| |z_2| = (|z_1| + |z_2|)^2
\]

\[\leq |z_1 z_2| = |z_1||z_2|\]

Taking the square root:

\[|z_1 + z_2| \leq |z_1| + |z_2| \quad \Rightarrow \quad \sum_{n=1}^{\infty} z_n \leq \sum_{n=1}^{\infty} |z_n|\]
Consider a complex sequence \( z_n \), \( n = 0, 1, \ldots, \infty \).

We define the complex series \( S_N = \sum_{n=0}^{N} z_n \).

This series is said to converge if \( S_N \to \) finite \( S \in \mathbb{C} \).

A simple criterion for convergence is the **ratio test**.

If \( \left| \frac{z_{n+1}}{z_n} \right| \to L < 1 \), then \( \sum_{n=0}^{\infty} z_n \) converges.

**Highlight of proof:**

Define \( n = L + 1 \) \( \Rightarrow \) \( L < n < 1 \).

\( \Rightarrow \) For \( n \) sufficiently large (say \( n > n_0 \)), \( \left| \frac{z_{n+1}}{z_n} \right| < L \)

\( \Rightarrow \left| \frac{z_{n+k}}{z_n} \right| = \left( \frac{z_{n+k}}{z_{n+k-1}} \right) \left( \frac{z_{n+k-1}}{z_{n+k-2}} \right) \cdots \left( \frac{z_{n+1}}{z_n} \right) < L^k \)

\( \Rightarrow \sum_{n=n_0}^{N} \left| \frac{z_{n+k}}{z_n} \right| < L^k \left( \sum_{n=n_0}^{N} \frac{1}{L^n} \right) \)

\( \therefore \sum_{n=n_0}^{N} z_n \leq \sum_{n=n_0}^{N} \left| z_n \right| = \sum_{n=n_0}^{N} \left| z_{n+k} \right| < L^k \sum_{n=n_0}^{N} \frac{1}{L^n} \)

\( P \quad N_0 > n_0 \)

\( n_0 \to \infty \quad \Rightarrow \quad \left| \frac{z_{n+k}}{z_n} \right| \to 0 \quad \sum_{n=n_0}^{\infty} \frac{1}{L^n} \to 0 \)

\( \Rightarrow \sum_{n=n_0}^{N} z_n \to 0 \quad \text{(geometric series)} \)

Given that \( L < 1 \), \( z_{n-N_0+1} \to 0 \) \( n \to \infty \).

This implies that \( \sum_{n=n_0}^{N} z_n \) is bounded by a convergent sequence as \( n \to \infty \).

\( \Rightarrow \) It cannot "blow up". \( \sum_{n=n_0}^{\infty} z_n \) exists and is s.t. \( \left| \sum_{n=n_0}^{\infty} z_n \right| < \frac{\left| z_{n_0} \right|}{1-L} \).

But \( \left| z_{n_0} \right| < \left| z_{n_0} \right| n_0^{-n_0} \to 0 \)

\( \Rightarrow \sum_{n=n_0}^{\infty} z_n \to 0 \), which means that \( \sum_{n=0}^{\infty} z_n \to \) finite number.

Conversely, if \( \left| \frac{z_{n+1}}{z_n} \right| \to L > 1 \), the series diverges (it means that \( |z_n| \) become bigger and bigger as \( n \to \infty \)).
Example: \( S_n = \sum_{n=0}^{\infty} a^n = \frac{1-a^{n+1}}{1-a} \), where \( a \in \mathbb{C} \) \( \{ i.e. \ z_n = a^n \} \)

- If \( |a| < 1 \) \( \Rightarrow a^{n+1} \to 0 \) \( \Rightarrow S_n \to \frac{1}{1-a} \)

- If \( |a| > 1 \) \( a^{n+1} \to \infty \) \( \Rightarrow S_n \to \infty \)

"\( \infty \)" is the complex \( \infty \) (Not just \( \pm \infty \)).

Ex: Suppose \( a = |a| e^{i\theta} \) \( a^n = |a|^n e^{i n \theta} \)

\( a^n \) acquires a larger and larger modulus, and rotates in the complex plane.

What if \( a = e^{i\theta} \), so \( |a| = 1 \) \( (\theta \pm 2\pi) \) \( \Rightarrow S_n = \frac{1-e^{i(n+1)\theta}}{1-e^{i\theta}} \)

This is always a finite number, but doesn't have a limit for \( n \to \infty \).

Conclusion: \( \sum_{n=0}^{\infty} a^n = \begin{cases} \frac{1}{1-a} & \text{if } |a| < 1 \\ \infty & \text{if } |a| > 1 \\ \text{undefined} & \text{if } |a| = 1 \end{cases} \)

**Definition: Taylor series, radius of convergence.**

Consider a function \( f(z) \) analytic over some part of \( \mathbb{C} \), containing \( z_0 \).

The Taylor series of \( f \) around \( z_0 \) in the series \( f(z) = \sum_{n=0}^{\infty} c_n (z-z_0)^n \)

If it only converges over a disk \( |z-z_0| < R \), then \( R \) is called the radius of convergence.
Examples.

Consider the function \( f(z) = \frac{1}{1-z} \).

It is analytic everywhere in \( \mathbb{C} \) except at \( z = 1 \).

We can write its Taylor expansion around \( z = 0 \):
\[
\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n.
\]

This is only valid for \( |z| < 1 \) (even though \( \frac{1}{1-z} \) is perfectly defined!)

One says that the radius of convergence of the Taylor series of \( f(z) = \frac{1}{1-z} \) around \( z = 0 \) is 1.

Consider \( g(z) = \frac{1}{z-a} \), \( a \in \mathbb{C} \).

This function is analytic everywhere except at \( z = a \).

Let us find its Taylor series around \( z_0 \neq a \):
\[
g(z) = \frac{1}{z-a} = \frac{1}{z-z_0-(a-z_0)} = \frac{1}{a-z_0} \frac{1}{1-\frac{z-z_0}{a-z_0}}.
\]

We already know that we can expand this if \( |\frac{z-z_0}{a-z_0}| < 1 \).

In that case, \( g(z) = \sum_{n=0}^{\infty} \frac{(z-z_0)^n}{(a-z_0)^{n+1}} \).

\( \Rightarrow \) the coefficients of the Taylor series are \( c_n = -\frac{1}{(a-z_0)^{n+1}} \).

the radius of convergence at \( z_0 \) is \( |a-z_0| \).

Question. What is the radius of convergence of \( f(z) = e^z \) around \( z = 0 \)?
\( \Rightarrow R = \infty \). The Taylor series \( e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \) is defined for all \( z \).
The Taylor series of the first term around \( z_0 \) has radius of convergence \( |z_0-a| \). The second one has radius of convergence \( |z_0-b| \).

They both converge if \( |z-z_0| < \min(|z_0-a|, |z_0-b|) \).

Let's show this from the Cauchy integral formula.

Take a contour \( \gamma \) inside of which \( f \) is analytic, containing \( z_0 \), and inside the region of convergence of the Taylor series \( f(z) = \sum_{n=0}^{\infty} c_n (z-z_0)^n \).

\[
\frac{d^n f}{dz^n}(z_0) = \frac{n!}{2\pi i} \oint_{\gamma} \frac{f(z)}{(z-z_0)^{n+1}} \, dz = \frac{n!}{2\pi i} \sum_{p=0}^{\infty} \oint_{\gamma} \frac{c_p (z-z_0)^p}{(z-z_0)^{n+1-p}} \, dz
\]

For \( p > n+1 \), the function \( (z-z_0)^{p-n-1} \) is analytic everywhere inside \( \gamma \).

\[
\oint_{\gamma} \frac{dz}{(z-z_0)^{p-n-1}} = 0 \quad \text{for} \quad p > n+1.
\]

To compute \( \oint_{\gamma} \frac{dz}{(z-z_0)^{q+1}} \), \( q > 0 \), apply Cauchy integral formula to \( g(z) = 1 \).

\[
\oint_{\gamma} \frac{dz}{(z-z_0)^{q+1}} = \frac{2\pi i}{q!} \frac{d^q g}{dz^q}igg|_{z_0} = \frac{2\pi i}{q!} \sum_{p=0}^{\infty} c_p 2\pi i \oint_{\gamma} \frac{dz}{(z-z_0)^{p+1-n}} = n! c_n \quad \text{for} \quad n \geq q \quad \text{non-zero}.
\]

\[
\frac{d^n f}{dz^n}(z_0) = \frac{n! \sum_{p=0}^{\infty} c_p 2\pi i \oint_{\gamma} \frac{dz}{(z-z_0)^{p+1-n}}}{2\pi i (n-r)!}
\]
Summary: The Taylor series of an analytic function around \( z = z_0 \) can always be written as
\[
f(z) = \sum_{n=0}^{\infty} \frac{(z-z_0)^n}{n!} \frac{d^n f}{dz^n}.\]

It is sometimes only defined within a finite region of convergence \( |z-z_0| < R \), where \( R \) is the radius of convergence at \( z_0 \).
(In general \( R \) depends on \( z_0 \)).

**Complex Laurent Series**

A Complex Laurent series takes the form
\[
f(z) = \sum_{n=-\infty}^{\infty} C_n (z-z_0)^n = \sum_{p=1}^{\infty} C_p \frac{1}{(z-z_0)^p} + \sum_{m=0}^{\infty} C_m (z-z_0)^m
\]

Negative powers \( f(z) \)

In general such a series has an annulus of convergence \( \mathrm{Re} \) \( |z-z_0| < R_{\text{max}} \)
(such as diverges outside this annulus).

**Example:** \( f(z) = \frac{1}{z-a} \quad z \neq a \)
\[
f(z) = \frac{1}{z-z_0} \quad \frac{1}{z-z_0 - (a-z_0)} = \frac{1}{z-z_0} \frac{1}{1 - \frac{a-z_0}{z-z_0}}
\]

If \( |a-z_0| < 1 \) \( \Rightarrow f(z) = \frac{1}{z-z_0} \sum_{n=0}^{\infty} \left(\frac{a-z_0}{z-z_0}\right)^n = \sum_{p=1}^{\infty} \frac{(a-z_0)^{p-1}}{(z-z_0)^p}
\]

\( |a-z_0| > 1 \quad z < 0 \infty \)

We had already seen that \( \text{for } |z-z_0| < |a-z_0| \), \( f(z) = -\sum_{n=0}^{\infty} \frac{(z-z_0)^n}{(a-z_0)^{n+1}} \)
\[ \frac{1}{z-a} = \sum_{p=1}^{\infty} \frac{(a-z_0)^{p-1}}{(z-z_0)^p} \quad \text{(negative powers)} \]
\[ \frac{1}{z-a} = -\sum_{n=0}^{\infty} \frac{(z-z_0)^n}{(a-z_0)^{n+1}} \quad \text{(positive powers)} \]

**Example (2)**. \( f(z) = \frac{1}{z-a} + \frac{1}{z-b} \), \( z_0 \neq a, b \).

\[ f(z) = \sum_{p=1}^{\infty} \frac{(a-z_0)^{p-1} + (b-z_0)^{p-1}}{(z-z_0)^p} \]
\[ f(z) = \sum_{p=1}^{\infty} \frac{(a-z_0)^{p-1}}{(z-z_0)^p} - \sum_{n=0}^{\infty} \frac{(z-z_0)^n}{(b-z_0)^{n+1}} \]
\[ f(z) = -\sum_{n=0}^{\infty} \left[ \frac{1}{(a-z_0)^{n+1}} + \frac{1}{(b-z_0)^{n+1}} \right] (z-z_0)^n \]

Three different Laurent/Taylor expansions depending on the region.

**Note**: While Taylor series are only defined around points where \( f \) is defined, we can define Laurent series around points where \( f \) is not defined. E.g., \( f(z) = \frac{1}{z-a} \) is its own Laurent series around \( z = a \), which is a singularity of \( f \).
Consider a function with a singularity at \( z = a \) (meaning, \( f \) is not defined at \( z = a \)).

If the most negative term in the Laurent series of \( f \) around \( a \) is \(-m\), the singularity is called a pole of order \( m \).

**Examples.**

\( f(z) = \frac{1}{z-a} \) has a pole of order 1 at \( z = a \).

\( f(z) = (z-a)^2 - \frac{2}{(z-a)^3} + \frac{17}{(z-a)^5} \) has a pole of order 5 at \( z = a \).

Without an explicit Laurent series, if \((z-a)^m f(z) \rightarrow \text{constant} \neq 0 \quad z \to a\) \( \Rightarrow f \) has a pole of order \( m \) at \( z = a \).

**Example.**

Consider \( f(z) = \frac{z}{\cos z - 1} \).

\[
\cos z = 1 - \frac{z^2}{2} + \cdots \quad \Rightarrow f(z) = \frac{z}{\cos z - 1} = \frac{2}{z^2 + O(z^4)} = \frac{2}{z^2} + O(z^3)
\]

\( \Rightarrow z f(z) = \frac{2}{1 + O(z^2)} \quad z \to 0 \quad \Rightarrow f \) has a pole of order 1 at \( z = 0 \).

**Removable singularities**

\( f(z) = \frac{\sin z}{z} \) appears to be singular at \( z = 0 \) (a priori not defined at 0)

But \( \sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \), with infinite radius of convergence.

\( \Rightarrow \frac{\sin z}{z} = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n+1)!} \) all positive

\( \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n+1)!} \)

Can be extended without problem at \( z \to 0 \).

Define \( f(0) \equiv 1 = \lim_{z \to 0} \frac{\sin z}{z} \).

This extended function is fully analytic on all \( C \).
• Essential singularities

If $\forall m > 0, (z-a)^m f(z) \to \infty$ as $z \to a$

Example: $f(z) = e^{1/z}$ we know that $rac{e^x}{x^m} \to \infty$ as $m \to \infty$

$\Rightarrow z e^{1/z} \to \infty$ as $z \to 0$

$\Rightarrow e^{1/z}$ has an essential singularity at $z = 0$.

• Residue Theorem

• Prelim: we saw at the beginning of lecture that for $z_0$ inside $E$:

$$\oint_{E} dz \frac{1}{(z-z_0)^n} = \begin{cases} 2\pi i & \text{if } n = -1 \\ 0 & \text{otherwise} \end{cases}$$

Consider a function $f(z)$ with a Laurent series at $z_0$:

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-z_0)^n.$$ 

Consider a contour $E$ inside annulus of convergence.

The $n = -1$ coefficient $c_{-1}$ is called the residue of $f$ at $z_0$.

$$\Rightarrow \oint_{E} dz \ f(z) = 2\pi i \ Residue(z_0)$$

• Suppose we have a function analytic on $C \setminus \{a,b\}$ (i.e. it has two poles at $z = a, b$). Take a contour encircling $a$ and $b$. 

$$\int_{C} dz \ f(z) = 2\pi i \ Residue(z_0)$$
Split the contour:

\[ \oint_{\gamma} f \, dz = \oint_{\gamma_1} f + \oint_{\gamma_2} = 2\pi i \left[ \text{Residue}(a) + \text{Residue}(b) \right]. \]

Now generally:

\[ \oint_{\gamma} f(z) = 2\pi i \sum \text{residues of poles inside } \gamma. \]

**Calculating the residue at a pole \( a \):**

- **If \( a \) is a pole of order 1:**
  \[ f(z) = \frac{c_1}{(z-a)} + c_0 + c_1(z-a) + \ldots \]
  \[ \Rightarrow c_1 = \lim_{z \to a} (z-a)f(z). \]

  **Example:**
  \[ f(z) = \frac{\sin z}{(z-a)(z-b)} \Rightarrow \text{Res}(a) = \lim_{z \to a} \frac{f(z)}{(z-a)} = \frac{\sin a}{a-b}. \]

- **If \( a \) is a pole of order 2:**
  \[ f(z) = \frac{c_2}{(z-a)^2} + \frac{c_1}{(z-a)} + c_0 + c_1(z-a) + \ldots \]
  \[ (z-a)^2 f(z) = c_2 + c_1(z-a) + c_0(z-a)^2 + \ldots \]
  \[ \frac{d}{dz} (z-a)^2 f(z) = c_1 + 2c_0 (z-a) \to c_1. \]

  **In general:** if \( a \) is a pole of order \( n \):
  \[ c_n = \lim_{z \to a} \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} \left[ (z-a)^n f(z) \right]. \]