Differentiations

• Terminology

- We will consider eqns of the form \( \frac{df}{dx} = f(x, y(x)) \)

Sometimes \( y \) is called the "dependent variable". Fundamentally, \( y \) is a function, not a variable. In contrast, \( x, y \) are called "independent variables" - they are the variables on which \( f \) depends.

\[
\frac{dy}{dx} = (y(x) - 3)^2 \quad \text{[dependent depends on independent]}
\]

- Ordinary differential equation (ODE): Has only one independent var.

- Partial differential equation (PDE): 2 or more independent vars.

\[
\begin{align*}
e.g.: \quad \frac{d y^3}{dx} + (y(x))^3 &= 3 \quad \text{ODE} \\
\frac{\partial y(x, y)}{\partial x \partial y} - \frac{3}{y(x, y)} &= 2 \quad \text{PDE}
\end{align*}
\]

- Linear differential equations: depend linearly on the dependent variable (i.e. function) and all its derivatives.

If not, it is a non-linear differential eqn.

\[
\begin{align*}
e.g.: \quad \frac{\partial (x, y)}{\partial x} + x^2y + \frac{\partial (x, y)}{\partial x} - \frac{3}{x^2+y^2} y(x, y) &= 0 \quad \text{linear} \\
\frac{d y}{dx} + 4(y(x))^2 &= 0 \quad \text{non-linear} \quad \text{(usually a lot harder to solve)}
\end{align*}
\]
The order of an ODE or PDE is the order of the highest derivative (whether ordinary or partial).

\[ \frac{\partial^3 f(x,y)}{\partial x^2 \partial y} + 4 \sin(f(x,y)) = 0 \quad : \quad 3 \text{-rd order} \]

\[ \frac{dl}{dx} + \frac{4}{l(x)} = 2 \quad : \quad 1 \text{-st order} \]

Order determines number of boundary conditions needed to solve.

Coupled differential equations have more than one dependent var.

\[ \frac{d^2 l_1}{dx^2} = l_1(x) - l_2(x), \quad \frac{d^2 l_2}{dx^2} = l_2(x) - l_1(x). \]

A homogeneous linear ODE or PDE only has terms proportional to the dependent variables.

\[ \frac{\partial^3 l}{\partial x \partial y} + 3 l(x,y) = 0 \quad \text{homogeneous} \]

\[ \frac{\partial^3 l}{\partial x \partial y} + 3 l(x,y) = \sin x \cos y \quad \text{non-homogeneous} \]

A homogeneous ODE/PDE always has \( l = 0 \) as a solution.

A non-homogeneous equation is also called driven: describe the response of a system to a driving force.

In general, we write a linear equation in the form

\[ L[l(x,y)] = S(x,y) \quad \text{Source (non-zero for non-homogeneous equations)} \]

linear differential operator

\[ L = \frac{\partial^3}{\partial x^2 \partial y} \]

\[ L[\lambda_1 l_1 + \lambda_2 l_2] = \lambda_1 L[l_1] + \lambda_2 L[l_2] \]
• If two functions are solutions to a homogeneous linear differential equation, then any linear combination is a solution as well:
\[ L[f_1(x)](x) = 0 \quad \text{and} \quad L[f_2(x)](x) = 0 \Rightarrow L[\lambda f_1(x) + \lambda f_2(x)](x) = 0. \]

• If a function \( f_p \) is a particular solution of a nonhomogeneous differential eq, then \( f_p + f_h \) is also a solution for any \( f_h \) that solves the homogeneous eq:
\[ L[f_h] = 0 \quad \text{and} \quad L[f_p] = S \]
\[ \Rightarrow L[f_h + f_p] = L[f_h] + L[f_p] = 0 + S = S. \]

**First-order ODEs**
(\( \Rightarrow \) need 1 boundary condition)

Consider the nonlinear, nonhomogeneous ODE:
\[ f''(x) + f(x) = \sin x \quad \text{with boundary condition} \quad f(0) = 0. \]

Recognize the left-hand side as \( \frac{1}{3} \frac{d}{dx} [f'(x)]^3 \)

right-hand side is \( -\frac{d}{dx} (\cos x) \quad \text{constant} \)

\[ \Rightarrow \quad \frac{1}{3} \frac{d}{dx} [f'(x)]^3 = -\frac{d}{dx} (\cos x) \Rightarrow \quad \frac{1}{3} [f'(x)]^3 = -\cos x + C \]

To determine \( C \): \( f(0) = 0 \Rightarrow \quad \frac{1}{3} = -1 + C \Rightarrow C = \frac{4}{3} \)

\[ \Rightarrow [f'(x)]^3 = 4 - 3 \cos x \quad \Rightarrow \quad f'(x) = (4 - 3 \cos x)^{1/3} \]

One can also, completely equivalently, use separation of variables.
\[ \frac{d^2 f}{dx^2} = \sin x \] multiply by \( df \) \[ \int_0^x f^2 \, df' = \int_0^x \sin x \, dx' \] (consistent with \( f(0) = 0 \))

\[ \Rightarrow \frac{1}{3} f^3 = 1 - \cos x \]

Some answer. Use whichever way makes more sense to you.

\[ \boxed{\frac{df}{dx} + P(x) \, f(x) = S(x), \quad f(x_0) = f_0.} \]

Solution to the linear, non-homogeneous, 1st-order ODE:

This method relies on two simple equations:

\[ \frac{d}{dx} e^x = e^x \]

\[ \frac{d}{dx} \left[ \int_{x_0}^{x} dx' \, P(x') \right] = P(x) \] (method called \( \text{integrating factor} \)).

\[ \frac{d}{dx} e^{g(x)} = e^{g(x)} \times \frac{dg}{dx} \] (chain rule).

\[ \Rightarrow \frac{d}{dx} \left( e^{g(x)} \, f(x) \right) = \frac{d}{dx} e^{g(x)} \times f(x) + e^{g(x)} \times \frac{df}{dx} \]

\[ \Rightarrow \frac{d}{dx} \left[ e^{g(x)} \, f(x) \right] = e^{g(x)} \left[ \frac{df}{dx} + \frac{dg}{dx} \times f(x) \right] \] (very useful equation).

\[ \ast \text{ Apply to } g(x) = \int_{x_0}^{x} dx' \, P(x') \] (\( x_0 \) fixed), \( \frac{dg}{dx} = P(x) \)

\[ \Rightarrow \frac{d}{dx} \left[ e^{\int_{x_0}^{x} dx' \, P(x')} \right] = e^{\int_{x_0}^{x} dx' \, P(x')} \left[ \frac{df}{dx} + \frac{dg}{dx} \times f(x) \right] \]
Let us now solve \( \frac{dl}{dx} + \lambda l(x) = S(l(x)) \) \( l(x) = b_0 \).

1. Multiply both sides by \( e^{\int_{x_0}^{x} d\eta' P(\eta')} \):

\[
e^{\int_{x_0}^{x} d\eta' P(\eta')} \left( \frac{dl}{dx} + \lambda l(x) S(l(x)) \right) = e^{\int_{x_0}^{x} d\eta' P(\eta')} S(l(x))
\]

2. Recognize the left-hand-side:

\[
\frac{d}{dx} \left[ e^{\int_{x_0}^{x} d\eta' P(\eta')} l(x) \right] = e^{\int_{x_0}^{x} d\eta' P(\eta')} S(l(x)) + \text{Constant}.
\]

3. Compute the anti-derivative:

\[
= b_0 \quad \text{so} \quad l\left(x_0\right) = b_0.
\]

4. Get rid of exponential on left-hand side:

\[
l(f) = b_0 e^{-\int_{x_0}^{x} d\eta' P(\eta')} + \int_{x_0}^{x} d\eta' S(l(\eta')) e^{\int_{x_0}^{x} d\eta' P(\eta')}.
\]

\[\boxed{\text{Simplest case: } \frac{dl}{dx} + \lambda l(x) = S(l(x)) \implies l(x) = \lambda \implies \int_{x_0}^{x} d\eta' P(\eta') = \lambda (\gamma - x_0)\]

\[
\implies l(\eta) = b_0 e^{-\lambda(\gamma - x_0)} + \int_{x_0}^{x} d\eta' S(l(\eta')) e^{-\lambda(\gamma - \eta')}
\]

\[\text{Example: Suppose we have a population of bacteria/radioactive elements/anything with a typical lifetime } \gamma.\]

\[\text{Call the number of bacteria/elements } N(t).\]
If it doesn't get renewed, \( \frac{dN}{dt} = -\frac{N(t)}{T} \Rightarrow N(t) = N(0) e^{-t/T} \). 

I.e. the initial population decays exponentially.

Now suppose we add new bacteria / elements at a rate \( S(t) \):

\[
\frac{dN}{dt} = \frac{-N(t)}{T} + S(t)
\]

\[
\Rightarrow N(t) = N(0) e^{-t/T} + \int_0^t S(t') e^{-\frac{t-t'}{T}} dt'
\]

\( \text{exponential decay between } t' \text{ and } t \).

Add \( S(t') dt' \) in time interval \( dt' \).

### Let's now solve example 10.2 of the textbook: (with a different method)

\[
\frac{dy}{dx} = -\frac{3x^2 + 2y^2}{4xy}
\]

\( y(1) = 1 \)

* Multiply by \( 2y \):

\[
\frac{dy}{dx} \frac{dy}{dx} = -\frac{3x^2 + 2y^2}{2x} \Rightarrow \frac{d}{dx} \left( \frac{y^2}{2} \right) = -\frac{3}{2} x
\]

Use the "integrating factor" method: multiply by \( e^{\int \frac{3}{2} dx} = e^{\frac{3}{2} x} \).

\[
\frac{d}{dx} \left( \frac{y^2}{2} \right) = -\frac{3}{2} x^3 + C
\]

\( y(1) = 1 \Rightarrow 1 = -\frac{3}{2} + C \Rightarrow C = \frac{3}{2} \)

\[
\Rightarrow y^2 = \frac{3}{2} - \frac{3}{2} x^3 \Rightarrow y^2 = \frac{3 - x^3}{2x} \Rightarrow y = \sqrt{\frac{3 - x^3}{2x}}
\]

### Second-order linear ODEs

**Example:** Harmonic oscillator: \( \frac{d^2 y}{dx^2} = -\omega^2 y(x) \)

\( \Rightarrow y(x) = A \sin(\omega x) + B \cos(\omega x) \)

(you should know the solution to this ODE).
Method of solution if \[ a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + cy = 0, \quad a, b, c \text{ real const.}, \quad a \neq 0. \]

1) Find the (complex) solutions of \[ aX^2 + bX + c = 0. \]

\[ X = \lambda_1, \lambda_2 \implies aX^2 + bX + c = a(X - \lambda_1)(X - \lambda_2) \]

with \( a(\lambda_1 + \lambda_2) = b, \quad a\lambda_1\lambda_2 = c. \)

Since \( a, b, c \) are real, we find \( \text{Im}(\lambda_1) = -\text{Im}(\lambda_2) \).

2) \[ a \left( \frac{d}{dx} - \lambda_1 \right) \left( \frac{dy}{dx} - \lambda_2 y \right) = a \left( \frac{d^2 y}{dx^2} - (\lambda_1 + \lambda_2) \frac{dy}{dx} + \lambda_1 \lambda_2 y \right) \]

\[ = a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + cy = 0. \]

Let's solve in two steps:

- Define \( s(x) = \frac{dy}{dx} - \lambda_2 y(x) \)

\[ \Rightarrow \frac{ds}{dx} - \lambda_1 s(x) = 0. \]

Linear, 1st-order homogeneous eq. for \( s(x) \).

- Simple solution: \( s(x) = A e^{\lambda_1 x} \)

- \( \frac{dy}{dx} - \lambda_2 y(x) = A e^{\lambda_1 x} \left( e^{-\lambda_2 x} \right) \)

- \( e^{-\lambda_2 x} \left( \frac{dy}{dx} - \lambda_2 y(x) \right) = A \left( e^{(\lambda_1 - \lambda_2)x} \right) \) \( \Rightarrow \) left-hand side \( = \frac{d}{dx} \left( e^{-\lambda_2 x} y(x) \right) \).

2. Case: \( \text{If } b^2 = 4ac \implies \lambda_1 = \lambda_2 = \lambda = -\frac{b}{2a}, \) a real number.

\[ \Rightarrow \frac{d}{dx} \left( e^{-\lambda x} y(x) \right) = A \implies e^{-\lambda x} y(x) = Ax + B \]

\[ \Rightarrow y(x) = e^{\lambda x} (Ax + B), \quad A, B \text{ determined from boundary conditions.} \]
\[ \text{If } \lambda_1 + \lambda_2 : \quad e^{-\lambda_2 x} y(0) = \frac{A}{\lambda_1 - \lambda_2} e^{(\lambda_1 - \lambda_2) x} + B \]

\[ \Rightarrow y(0) = A e^{\lambda_1 x} + B e^{\lambda_2 x} \]

\[ \Rightarrow \]

\[ \text{If } b^2 - 4ac > 0, \quad \lambda_1, \lambda_2 \text{ both real, } A, B \text{ both real, determined from boundary conditions.} \]

\[ \Rightarrow \text{If } b^2 - 4ac < 0, \quad \lambda_1 = -\frac{b}{2a} + i \sqrt{b^2 - 4ac} = \lambda_{Re} + i \lambda_{Im} \]

\[ \lambda_2 = \lambda_{Re} - i \lambda_{Im}, \]

\[ \text{so } y(0) = e^{\lambda_{Re} x} \left( A e^{i \lambda_{Im} x} + B e^{-i \lambda_{Im} x} \right). \quad A, B \text{ complex.} \]

We can rewrite this as a linear combination:

\[ y(0) = e^{\lambda_{Re} x} \left( \alpha \cos(\lambda_{Im} x) + \beta \sin(\lambda_{Im} x) \right), \quad \alpha, \beta \text{ real.} \]

This is an example where complex numbers are a useful intermediate step, even though the final result is a real number.

Simplest application: \[ d^2 y + \omega^2 y = 0 \quad \lambda^2 + \omega^2 = 0 \quad \Rightarrow \lambda = \pm i \omega \]

\[ \Rightarrow y(x) = \alpha \cos(\omega x) + \beta \sin(\omega x). \]

**The Wronskian**

Most general linear, second-order, homogeneous ODE:

\[ y''(x) + P(x) y'(x) + Q(x) y(x) = 0 \]

\[ \left[ y' = \frac{dy}{dx}, \quad y'' = \frac{d^2 y}{dx^2} \right]. \]

Second-order \(\Rightarrow\) there should be 2 independent solutions: \(y_1(x), y_2(x)\)

Define \(W(x) = y_1(x) y_2'(x) - y_2(x) y_1'(x)\)

\[ W' = y_1 y_2'' + y_1' y_2' - y_2' y_1' - y_2 y_1'' = y_1 y_2'' - y_2 y_1''. \]
Now, both \( y_1 \) and \( y_2 \) are solutions \( \Rightarrow y_1'' = -P(x) y_1' - Q(x) y_1 \),
\( y_2'' = -P(x) y_2' - Q(x) y_2 \),
\( \Rightarrow W'(x) = -y_1(x)\left[Py_2' + Qy_2\right] + y_2\left(Py_1' + Qy_1\right) = P \left(-y_1 y_2' + y_2 y_1'\right) = -PW \).

\( \Rightarrow \) The Wronskian \( W(y) \) satisfies the first order ODE,
\[ \frac{dW}{dx} + P(x) W(y) = 0 \]
We know how to solve this: \( W(x) = W_0 e^{-\int_{x_0}^{x} P(x) \, dx} \).

- **Application I**: Find \( y_2 \) given \( y_1 \).

\( \Rightarrow \) Suppose we found \( y_1(x) \). We want to find \( y_2(x) \).
\[ \frac{1}{y_1} \frac{dy_2}{dx} = \frac{y_2'}{y_1} - \frac{y_1 y_2'}{y_1^2} = \frac{y_2 y_1' - y_1 y_2'}{y_1^2} = \frac{W}{y_1^2} \]
\( \Rightarrow \frac{y_2}{y_1}(x) = \int_{x_0}^{x} \frac{W(y)}{[y_1(y)]^2} \, dy + C \)
\( \Rightarrow y_2(x) = C y_1(x) + y_1(x) \int_{x_0}^{x} \frac{W(\nu)}{[y_1(\nu)]^2} \, d\nu \).
(can drop \( C \) if \( y_2 \) is independent of \( y_1 \)).

- **Application II**: Nonhomogeneous equations

\[ y''(x) + P(x) y'(x) + Q(x) y(x) = S(x) \]
Suppose we have 2 solutions of the homogeneous eq., \( y_1(x), y_2(x) \).
Define:
\[ z_1(x) = \int_{x_0}^{x} \frac{y_1(\nu)}{W(\nu)} S(\nu) \quad z_2(y) = \int_{x_0}^{x} \frac{y_2(\nu)}{W(\nu)} S(\nu) \]
\[ \Rightarrow z_1'(x) = \frac{y_1(x)}{w(x)} S(x), \quad z_2'(x) = \frac{y_2(x)}{w(x)} S(x). \]

Define \( y_p(x) = y_2(x) z_1(x) - y_1(x) z_2(x) \)

\[ \Rightarrow y_p' = y_2' z_1 + y_2 \frac{y_1}{w} S - y_1' z_2 - y_1 \frac{y_2}{w} S = y_2' z_1 - y_1' z_2. \]

\[ y_p'' = y_2'' z_1 + y_2' \frac{y_1}{w} S - y_1'' z_2 - y_1' \frac{y_2}{w} S = y_2'' z_1 - y_1'' z_2 + S \]

\[ y_2' S - y_1' S = \frac{w}{w} S = S. \]

\[ \Rightarrow y_p'' + P y_p' + Q y_p = z_1(y_2'' + P y_1' + Q y_1) - z_2(y_1'' + P y_2' + Q y_2) + S \]

\[ = 0 \quad = 0 \]

\( y_1 \) and \( y_2 \) solutions of homogeneous eq.

\[ \Rightarrow y_p \text{ is a particular solution of the nonhomogeneous equation.} \]

**Note:** You should NOT remember this by heart. It is useful to remember that a particular solution can be written in terms of the Wronskian.

- **Method of Frobenius**

  Series approach method to solve linear, homogeneous ODEs.

  We will study a specific example: \( x^2 y''(x) + x y'(x) + x^2 y(x) = 0 \)

  Look for a solution of the form \( y(x) = \sum_{n=0}^{\infty} c_n x^n \)

  \[ \sum_n c_n \left( n(n-1) x^n + n x^n + x^{n+2} \right) = 0 \]

  Define \( p = n+2 \) dummy variable

  \[ \Rightarrow n = p-2 \]

  \[ \Rightarrow \sum_n \left[ (n(n-1) + n) c_n + c_{n-2} y x^n \right] = 0 \quad \text{(renamed } p \rightarrow n). \]
Can only hold for all \( x \) if \( \mu c_n + c_{n-2} = 0 \).

This gives us a recursion relation. Implies \( c_0 = 0 \), \( c_0 \) undetermined.

\[ (-2)^n c_{-\ell} + c_{-\ell-2} = 0 \Rightarrow c_{-\ell} = 0 \text{, etc.} \quad c_{-2\ell} = 0, \quad \forall \ell > 0. \]

* For positive even \( n = 2p \):

\[
C_{2p} = -\frac{C_{2(p-1)}}{(2p)^2} = + \frac{C_{2(p-2)}}{4^2 \left[p(p-1)\right]^2} = - \frac{C_{2(p-1)}}{4^3 \left[p(p-1)(p-2)\right]^2} = (-1)^{p} \frac{c_0}{4^p (p!)^2}
\]

* All odd \( n = 2p+1 \) are related to \( c_1 \):

\[
C_{2p+1} = -\frac{C_{2(p-1)+1}}{(2p+1)^2} = + \frac{C_{2(p-1)+1}}{(2p+1)(2p-1)!} = (-1)^{p} \frac{c_1}{(2p+1)!}
\]

\[ c_{-1} = -c_1 \text{. Can then relate all } c_{-(2p+1)} \text{ to } c_{-1} = -c_1. \]

**Bottom line:** \( y(x) = c_0 \sum_{p=0}^{\infty} (-1)^{p} \frac{x^{2p}}{4^p (p!)^2} + c_1 \sum_{p=0}^{\infty} x^{2p+1} \)

Two independent solutions, defined as series.

For this specific example, these are \( j_0, y_0 \), spherical Bessel function of the first and second kind.

**Radius of convergence:** Use ratio test.

\[
Z_p \equiv (-1)^{p} \frac{x^{2p}}{4^p (p!)^2} \quad j_0(x) = Z_0 \sum_{p=0}^{\infty} \left| Z_p \right| = \frac{x^2}{4 (p+1)^2} \xrightarrow[p \to \infty]{} 0
\]

\( \Rightarrow \) This series has an infinite radius of convergence.
Method of quadrature

Suppose we want to solve \( y''(x) = f(y(x)) \):

* Multiply both sides by \( y'(x) \):
  \( y'(x) y''(x) = f(y(x)) y'(x) \)

* Recognize that \( \frac{d}{dy} \left( \frac{1}{2} [y'(x)]^2 \right) = y'(x) y''(x) \).

* Find the antiderivative of \( f \):
  \( g(y) \) s.t. \( \frac{dg}{dy} = f \).

\[ \Rightarrow \frac{d}{dx} \left( g(y(x)) \right) = \frac{dg}{dy} \cdot y'(x) = f(y(x)) y'(x) \quad \text{[chain rule]} \]

\[ \Rightarrow \text{The equation can then be rewritten as} \quad \frac{d}{dx} \left( \frac{1}{2} [y'(x)]^2 \right) = 2 \frac{d}{dx} \left[ g(y(x)) \right] \]

\[ \Rightarrow \left( y'(x) \right)^2 = 2 g(y(x)) + C \quad \text{determined by boundary conditions} \]

\[ \Rightarrow y'(x) = \pm \sqrt{2 g(y(x)) + C} = \frac{dy}{dx} \]

Now we integrate by method of separation of variables:

\[ \frac{dy}{\sqrt{2 g(y) + C}} = \pm dx \quad \Rightarrow \int_{y_0}^{y} \frac{dy}{\sqrt{2 g(y) + C}} = x - x_0 \quad \text{determined by 2nd boundary condition} \]

**Examples.** (i) \( y''(x) = \frac{1}{\sqrt{y(x)}} \quad y(0) = 0 \quad y'(0) = 0 \)

\[ f(y) = \frac{1}{\sqrt{y}} \quad \text{Antiderivative:} \quad g(y) = 2 \sqrt{y} \]

Multiply by \( g'(y) \):

\[ y'(x) y''(x) = \frac{y'(x)}{\sqrt{y(x)}} \]

\[ \Rightarrow \frac{d}{dx} \left[ \frac{1}{2} [y'(x)]^2 \right] = \frac{d}{dx} \left[ 2 \sqrt{y(x)} \right] \]
\[ \frac{1}{2} \left[ y'(b) \right]^2 = 2 \sqrt{y(x)} + C \]

Plug in boundary conditions at \( x = 0 \Rightarrow C = 0 \)

\[ \Rightarrow \left[ y'(b) \right]^2 = 4 \sqrt{y(b)} \]

\[ \Rightarrow y'(b) = \pm 2 \left[ y(b) \right]^{rac{1}{4}} \]

Separation of variables: \( \frac{dy}{y^{rac{3}{4}}} = \pm \frac{2}{x} \, dx \)

Integrate, using \( y(0) = 0 \)

\[ \int_0^y \frac{dy'}{y'^{\frac{1}{4}}} = \pm \int_0^x 2 \, dx \Rightarrow \frac{4}{3} y^{\frac{3}{4}} = \pm 2x \]

The only real solution is with the plus sign: \( y(x) = \left( \frac{3}{2}x \right)^{\frac{4}{3}} \)

It is always good to check. \( y'(b) = \frac{4}{3} \left( \frac{3}{2}x \right)^{\frac{1}{3}} \times \frac{3}{2} = 2 \left( \frac{3}{2}x \right)^{\frac{1}{3}} \)

\[ \left( \text{Chain rule!} \right) \]

\[ y''(x) = 2 \cdot \frac{1}{3} \left( \frac{3}{2}x \right)^{-\frac{1}{3}} \times \frac{3}{2} = \left( \frac{3}{2}x \right)^{-\frac{2}{3}} = \frac{1}{\sqrt{y(x)}} \]

\[ (ii) \quad R''(r) = -\frac{GM}{R(r)^2} \quad R(0) = R_0, \quad R'(0) = 0 \]

\[ g(r) = -\frac{GM}{R^2} \quad \text{Anti-derivative:} \quad g(r) = \frac{GM}{R} \]

Multiply by \( R'(r) \Rightarrow \quad R'(r) R''(r) = -\frac{GM}{R(r)^2} R'(r) \)

\[ \frac{d}{dr} \left( \frac{1}{2} R'(r)^2 \right) = \frac{d}{dr} \left( \frac{GM}{R(r)} \right) \]

\[ \Rightarrow \frac{1}{2} R'(r)^2 - \frac{GM}{R(r)} = C \quad \text{Plug in:} \quad R(0) = R_0, \quad R'(0) = 0 \]

\[ \Rightarrow C = -\frac{GM}{R_0} \]

Physical interpretation: conservation of energy

\[ \Rightarrow R'(r) = \pm \sqrt{2GM} \left( \frac{1}{R(r)} - \frac{1}{R_0} \right)^{\frac{1}{2}} \]
This only makes sense if \( R(t) < R_0 \). Initially, \( R'(t) < 0 \).

Use separation of variables:
\[
\frac{dR}{(\frac{1}{R} - \frac{1}{R_0})^2} = -\sqrt{2GM} \ dt
\]

Integrate:
\[
\int_{R_0}^{R} \frac{dR'}{(\frac{1}{R'} - \frac{1}{R_0})^2} = -\sqrt{2GM} (R - R_0)
\]

Define \( x = \frac{R}{R_0} \), \( dx = \frac{R}{R_0^2} \ dt \) \Rightarrow
\[
\int_{\frac{1}{R_0}}^{\frac{1}{R}} \frac{dx}{(\frac{1}{x} - 1)^2} = \sqrt{\frac{2GM}{R_0^3}} (R - R_0)
\]

This does have some explicit expression, although awful, of the form:
\[
F(\frac{R}{R_0}) = \sqrt{\frac{2GM}{R_0^3}} (R - R_0), \quad F(2) = \int_{\frac{1}{R_0}}^{2} \frac{dx}{(\frac{1}{x} - 1)^2}
\]

We can find \( R(t) \) upon inverting this function (numerically).

Green's Function

Consider a linear nonhomogeneous order \( n \) ODE of the form:
\[
\mathcal{L}[y](x) = S(x)
\]
where \( \mathcal{L}[y] \) is a linear differential operator.

Eg: For second-order ODE, \( \mathcal{L}[y](x) = \frac{d^2}{dx^2} y + P(x) \frac{dy}{dx} + Q(x) y \).

Suppose we moreover have homogeneous boundary conditions, i.e. some linear combinations of all first derivatives = 0.
For a second order ODE,
\[2f'(a) + 3f(a) = 0 \quad \text{boundary condition at } x = a\]
\[f'(b) - 2f(b) = 0 \quad \text{boundary condition at } x = b.\]

The basic idea of a Greens function comes from the principle of superposition:

Suppose \( S(x) = \lambda_1 S_1(x) + \lambda_2 S_2(x) \)
\[L[f_1](x) = S_1(x), \quad L[f_2](x) = S_2(x)\]
+ \( f_1, f_2 \) satisfy homogeneous boundary conditions.

\[\Rightarrow \text{Define } f = \lambda_1 f_1 + \lambda_2 f_2, \quad L[f](x) = S(x) \quad \text{for any } \lambda \]
\( f \) satisfies boundary conditions since they are homogeneous.

Now suppose we have \( S(x) = \sum_{n=0}^{\infty} \lambda_n S_n(x) \)
\[L[f_n](x) = S_n(x) \Rightarrow \text{solution } f(x) = \sum_{n=0}^{\infty} \lambda_n B_n(x)\]

Suppose \( S(x) = \int_{-\infty}^{+\infty} dy \lambda(y) R(x, y) \)
Fix \( y, L \) concerns only \( x \)

Define \( F(x,y) \) s.t. \( L[F(x,y)] = R(x,y) \)
\[\Rightarrow f(x) = \int_{-\infty}^{+\infty} dy \lambda(y) F(x, y) \text{ is the solution of } L[f] = S.\]
Just think of \( \int_{-\infty}^{+\infty} dy \lambda(y) F(x, y) \approx \sum_{n} \lambda(y_n) F(x, y_n) \) for some particular \( y_n \).
Example: \[ \mathcal{L}[f](x) = \frac{d^2 f}{dx^2} + p(x) \frac{df}{dx} + q(x) f(x) = S(x) = \int dy \, R(x, y). \]
\[
\begin{align*}
& f'(0) = 0 \\
& f'(1) - 2f(1) = 0.
\end{align*}
\]

\[ F(x, y) \text{ satisfies } \begin{cases}
\frac{\partial^2 F}{\partial x^2} + p(x) \frac{\partial F}{\partial x} + q(x) F(x, y) = R(x, y), \\
\frac{\partial F}{\partial x}(1, y) = 0, \\
\frac{\partial F}{\partial x}(0, y) = 0, \\
\frac{\partial^2 F}{\partial x^2}(1, y) - 2F(1, y) = 0, \quad \forall y.
\end{cases} \]

- We can always write \( S(x) = \int_{-\infty}^{+\infty} dy \, S(y) \, \delta_D(x-y) \).

We define the Green's function \( G(x, y) \) s.t.
\[ \mathcal{L}[G(x, y)](x) = \delta_D(x-y) + \text{boundary conditions}. \]
\[ \left( \text{e.g. } \frac{\partial^2 G}{\partial x^2} + p(x) \frac{\partial G}{\partial x} + q(x) G(x, y) = \delta_D(x-y) \right) \]

\( \Rightarrow \) the full solution is \( f(x) = \int_{-\infty}^{+\infty} dy \, S(y) \, G(x, y) \).

**Why is this useful?**

Suppose you don't know in advance what \( S \) is going to be. If you compute the Green's function \( G(x, y) \), you will be able to find the solution for any \( S \).
We can actually solve this explicitly:
\[ y'(x) = y'(0) + \int_0^x F(x') \quad [x' = dummy\ integration\ variable] \]

Integrate again: \[ y(x) = y'(0) x + \int_0^x dx' \int_0^{x'} F(x'') \]

Let us simplify the double integral by changing the order of \( x', x'' \):

Current integral: integrate over \( x'' \in [0, x'] \)

\( \text{Then } x' \in [0, x] \).

Equivalent: integrate over \( x' \in [x'', x] \)

\( \text{Then } x'' \in [0, x'] \).

\[ y(x) = y'(0) x + \int_0^x dx'' \int_0^{x''} F(x'') \]

\[ = y'(0) x + \int_0^x dx'' (x-x'') F(x'') \]

We still don't know \( y'(0) \): we know that \( y(1) = 0 \)

\[ \Rightarrow \text{Plug } x = 1 \Rightarrow 0 = y'(0) + \int_0^1 dx'' (1-x'') F(x'') \]

So we found \[ y(x) = \int_0^x dx'' (x-x'') F(x'') - x \int_0^1 dx'' (1-x'') F(x'') \]
Summary of last lecture:

For a linear ODE, with homogeneous boundary conditions,

\[ L[y](x) = S(x), \]

the solution can be written as

\[ f(x) = \int dy \, G(x, y) \, S(y). \]

The function \( G(x, y) \) is the Green's function. For a given \( y \), it satisfies

\[ L[G(x, y)](x) = \delta(y-x), \]

with appropriate boundary conditions.

We started working on the example \( f''(x) = S(x), \quad f(0) = 0 = f(1) \)

We solved this explicitly and found (different notation)

\[ f(x) = \int_0^x dy \, (x-y) \, S(y) - x \int_0^1 dy \, (1-y) \, S(y). \]

Let us rewrite the first integral as

\[ \int_0^1 dy \, (x-y) S(y) \, H(x-y), \]

where \( H \) is the Heaviside function.

\[ \Rightarrow \text{We therefore found the Green's function:} \]

\[ f(x) = \int_0^1 dy \, G(x, y) \, S(y), \quad G(x, y) = (x-y) \, H(x-y) - x \, (1-y) \]

\[ \Rightarrow \text{For } x < y: \quad G(x, y) = -x \, (1-y) \]

\[ \text{For } x > y: \quad G(x, y) = x-y - x+y = -y \, (1-x) \]

\[ G(0, y) = G(1, y) = 0, \]

as it should.
Let's check that $G$ indeed solves the coned ODE:

\[ \frac{\partial G}{\partial x} = H(x-y) + (x-y) H'(x-y) - (1-y) = H(x-y) + (x-y) \delta_d(x-y) - (1-y) \]

Now $(x-y) \delta_d(x-y) = 0$ everywhere (including at $x=y$!) \[ \Rightarrow \frac{\partial G}{\partial x} = H(x-y) - (1-y) \]

Expressing: \[ \frac{\partial G}{\partial x} = \begin{cases} - (1-y) & [x<y] \\ y & [x>y] \end{cases} \]

This makes sense: $G(x,y)$ has a constant slope for $x<y$ and $x>y$.

\[ \frac{\partial^2 G}{\partial x^2} = H'(x-y) = \delta_d(x-y) \]

\[ \Rightarrow G \text{ satisfies } \mathcal{L}[G(x,y)]|_{x} = \delta_d(x-y) \text{, and boundary cond.} \]

Going the other way around:

We seek the function $G(x,y)$ that solves

\[ \frac{\partial^2 G}{\partial x^2} = \delta_d(x-y) \quad \Rightarrow G(0,y) = 0 = G(1,y) \]

For $x=y$, this is $\frac{\partial^2 G}{\partial x^2} = 0$.

\[ \Rightarrow G(x,y) = \begin{cases} A + B x & x<y \\ C + D x & x>y \end{cases} \]

To find the constants:

\[ G(0,y) = 0 \Rightarrow A = 0 \]
\[ G(1,y) = 0 \Rightarrow C + D = 0 \]

The constants $A, C$ and $B, D$ need not be the same!
So we have \( G(x,y) = \begin{cases} \frac{B}{x} & x < y \\ C(1-x) & x > y \end{cases} \) (1)

Go back to \( \frac{\partial^2 G}{\partial x^2} = \delta_D(x-y) \) and integrate between \( y-f \) and \( y+f \):

\[
\int_{y-f}^{y+f} dx \frac{\partial^2 G}{\partial x^2} = \int_{y-f}^{y+f} dx \delta_D(x-y) = 1
\]

\[
= \frac{\partial G(x,y+f,y)}{\partial x} - \frac{\partial G(x,y-f,y)}{\partial x}
\]

From (1), we have \( \frac{\partial G}{\partial x} = \begin{cases} \frac{B}{x} & x < y \\ -C & x > y \end{cases} \)

From the jump condition: \(-C - B = 1 \implies C = -(1+B)\)

Now, while \( \frac{\partial G}{\partial x} \) can have a jump (because \( \frac{\partial^2 G}{\partial x^2} \propto \delta_D \)), \( G \) itself must be continuous at \( x = y \). Otherwise \( \frac{\partial G}{\partial x} \) would be proportional to \( \delta_D(x-y) \) (and we know it is not!).

\[
\implies \text{ From (1), } BY = C(1-y)
\]

Solving \( \begin{cases} C = -(1+B) \\ BY = C(1-y) \end{cases} \implies \begin{cases} B = -(1-y) \\ C = -y \end{cases} \)

So we find \( G(x,y) = \begin{cases} -(1-y)x & x < y \\ -y(1-x) & x > y \end{cases} \)

This is indeed the same Green's function we had found earlier!