

Problem Set 1: Cosmological Parameters and Their Evolution  
Due Wednesday, February 18

Recall that:

the redshift  $z \equiv \frac{a_0}{a} - 1$

the Hubble parameter  $H \equiv \frac{\dot{a}}{a}$

the density parameter  $\Omega \equiv \frac{\rho}{\rho_c}$

where  $a_0$  is the value of the expansion factor at  $z = 0$ . In our standard notation (adopted below),  $a_0 \equiv 1$ , so  $z = a^{-1} - 1$ .

The Friedmann equation is

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2 R_0^2}, \quad k = -1, 0, \text{ or } +1, \quad (1)$$

where  $\rho$  is the *total* energy density. (1) Write the Friedmann equation in terms of  $H$  and  $\Omega$  (instead of  $\dot{a}$  and  $\rho$ ). What is the relation between the values of  $\Omega$  and  $k$ ? For  $k \neq 0$ , what is the value of the curvature radius  $R = aR_0$  in terms of  $H$  and  $\Omega$ ?

(2) Now assume that the energy components of the universe are matter, radiation, and a constant vacuum energy (i.e., a cosmological constant). The energy densities of these components scale with redshift as

$$\rho_m = \rho_{m,0}(1+z)^3$$

$$\rho_r = \rho_{r,0}(1+z)^4$$

$$\rho_\Lambda = \rho_{\Lambda,0} = \text{const.},$$

where subscripts zero denote values at  $z = 0$ . Define

$$\Omega_m = \frac{\rho_m}{\rho_c}, \quad \Omega_r = \frac{\rho_r}{\rho_c}, \quad \Omega_\Lambda = \frac{\rho_\Lambda}{\rho_c}$$

implying

$$\Omega = \Omega_m + \Omega_r + \Omega_\Lambda.$$

Further define

$$\Omega_k = -\frac{kc^2}{a^2 R_0^2 H^2}.$$

[Note that  $\Omega_k = -kD_H^2/(a^2 R_0^2)$  where  $D_H(z) \equiv cH^{-1}(z)$  is the Hubble distance.]

Use the Friedmann equation to show that

$$\Omega_m + \Omega_r + \Omega_\Lambda + \Omega_k = 1.$$

**Caution:** In the older cosmological literature,  $\Omega$  without subscript is often used to denote  $\Omega_m$  alone. The convention adopted here follows the now mostly standard but not quite universal trend to have  $\Omega$  refer to the contribution of all energy components and denote the matter density parameter  $\Omega_m$ .

(3) Show that the Friedmann equation can be written in the form

$$H(z) = H_0 \left[ \Omega_{\Lambda,0} + \Omega_{k,0}(1+z)^2 + \Omega_{m,0}(1+z)^3 + \Omega_{r,0}(1+z)^4 \right]^{1/2}, \quad (1)$$

where subscripts 0 refer to the present day values of parameters. (Hint: Draw on your solution from part (1) and think carefully about how  $\rho$  and  $\rho_c$  change with redshift.)

(4) Show that the lookback time to an object at redshift  $z$  (i.e., the elapsed time from redshift  $z$  to redshift 0) is

$$t = \frac{1}{H_0} \int_0^z \frac{dz'}{(1+z') [\Omega_{\Lambda,0} + \Omega_{k,0}(1+z')^2 + \Omega_{m,0}(1+z')^3 + \Omega_{r,0}(1+z')^4]^{1/2}}. \quad (2)$$

(5) Show that

$$\Omega_m(z) = \frac{\Omega_{m,0}(1+z)^3}{\Omega_{\Lambda,0} + \Omega_{k,0}(1+z)^2 + \Omega_{m,0}(1+z)^3 + \Omega_{r,0}(1+z)^4}. \quad (3)$$

(6) Assume that  $\Omega_{r,0}$  and  $\Omega_{\Lambda,0}$  are negligibly small (i.e., they can be set to zero in equation 3). What are the asymptotic values of  $\Omega_m$  and  $\Omega_k$  for  $z \gg \Omega_{m,0}^{-1}$ ?

Plot  $\Omega_m(z)$  in this case for  $k = +1, 0, -1$ . I prefer a schematic *hand drawn* plot to a computer-generated plot of the equation, but be sure to label axes, mark relevant values, and show all of the relevant features of the behavior.

(7) Assume instead that  $\Omega_{r,0}$  and  $\Omega_{k,0}$  are negligibly small, but  $\Omega_{\Lambda,0}$  is not. What are the asymptotic values of  $\Omega_m$  and  $\Omega_{\Lambda}$  for  $z \gg \Omega_{m,0}^{-1}$ ?

Make a schematic plot of  $\Omega_m(z)$  comparing the case  $\Omega_{m,0} = 0.3, \Omega_{k,0} = 0, \Omega_{\Lambda,0} = 0.7$  to the case  $\Omega_{m,0} = 0.3, \Omega_{k,0} = 0.7, \Omega_{\Lambda,0} = 0$ .

(8) The space density of bright galaxies (as bright as or brighter than the Milky Way) is about 0.01 galaxies/Mpc<sup>3</sup>. Several of the best current measurements of the Hubble constant yield  $H_0 \approx 70$  km/s/Mpc. Assume that (a) the above value for  $H_0$  is correct, (b) the average mass of a bright galaxy is  $2 \times 10^{12} M_{\odot}$  (including the dark matter halo), and (c) essentially all of the matter in the universe is attached to bright galaxies. What is the value of  $\Omega_{m,0}$ ?

[1 Mpc =  $3.09 \times 10^{24}$  cm.  $1 M_{\odot} = 1$  solar mass =  $2 \times 10^{33}$  g.]

(9) With the same assumptions, what is the ratio of the matter energy density  $\rho_{m,0}$  to the energy density  $\rho_{r,0}$  of the 2.73° radiation background? At what redshift were  $\rho_m$  and  $\rho_r$  equal?

(10) The “Planck density” is  $\rho_{\text{Planck}} = c^5/(\hbar G^2)$ , the only combination of the fundamental constants  $c$ ,  $G$ , and  $\hbar$  that has units of density. What is the Planck density in g cm<sup>-3</sup>? If  $\rho_{\Lambda,0}$  is equal to the Planck density and  $H_0 = 70$  km/s/Mpc, what is  $\Omega_{\Lambda,0}$ ?